Problem Set 3B: Practice Problem Set 3 – Solutions

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MAT 342 – Applied Complex Analysis

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NEVER DUE. Do the exercises for your own benefit. Practice makes perfect. On this note, keep in mind that the assignments are mostly for grading purposes and are thus not enough practice. If you have any questions, let me know.

Exercise 0. Review everything you've studied this week before proceeding!

Exercise 1. Evaluate $\int_\gamma \left(z - \frac{1}{z}\right) dz$ where γ is the straight-line path from 1 to i.

Solution: We may parametrize γ by $\gamma(t) = (1-t)\cdot 1 + t \cdot i = 1 + (i-1)t, 0 \le t \le 1$. Therefore:

$$\begin{aligned}
\int_\gamma \left(z - \frac{1}{z}\right) dz &= \int_0^1 \left(1 + (i-1)t - \frac{1}{1+(i-1)t}\right)(i-1)dt \\
&= (i-1)\int_0^1 (1+(i-1)t)dt - (i-1)\int_0^1 \frac{1}{1+(i-1)t}dt \\
&= (i-1)\left[t + (i-1)\frac{t^2}{2}\right]_{t=0}^{t=1} - (i-1)\left[\frac{\log(1+(i-1)t}{i-1}\right]_{t=0}^{t=1} \\
&= (i-1)\left(1 + (i-1)\frac{1}{2}\right) - \frac{i-1}{i-1}\Big(\big(\ln(|i|) + i\arg i\big) - \big(\ln(|1|) + i\arg 1\big)\Big) \\
&= (i-1) + \frac{(i-1)^2}{2} - (i\pi/2 - i(0)),
\end{aligned}$$

by choosing the branch $-\pi \le \arg(z) \le \pi$ for the log. The final result is $-1 - i\frac{\pi}{2}$.

Note that it does not matter which branch we choose because the factor of $i2\pi$ which will be added to the argument will cancel out in the evaluation.

Exercise 2. Compute $\int_C |z|\, dz$ where C is the rectangle with corners $-1, 1, 1+i$ and $-1+i$.

Solution: The rectangle with such corners can be represented as the sum of four line contours: C_1 from -1 to 1, C_2 from 1 to $1+i$, C_3 from $1+i$ to $-1+i$ and C_4 from $-1+i$ back to 1. Each one can be parametrized as follows.

- $C_1 : z_1(t) = t, -1 \leq t \leq 1$.
- $C_2 : z_2(t) = 1 + it, 0 \leq t \leq 1$.
- $C_3 : z_3(t) = -t + i, -1 \leq t \leq 1$.
- $C_4 : z_4(t) = -1 - it, -1 \leq t \leq 0$.

Therefore:

$$\begin{aligned}
\int_C |z|\, dz &= \int_{C_1} |z|\, dz + \int_{C_2} |z|\, dz + \int_{C_3} |z|\, dz + \int_{C_4} |z|\, dz \\
&= \int_{-1}^{1} \left|z_1(t)\right| z_1'(t)dt + \int_{0}^{1} \left|z_2(t)\right| z_2'(t)dt + \int_{-1}^{1} \left|z_3(t)\right| z_3'(t)dt + \int_{-1}^{0} \left|z_4(t)\right| z_4'(t)dt \\
&= \int_{-1}^{1} (|t|)(1)dt + \int_{0}^{1} (|1+it|)(i)dt + \int_{-1}^{1} (|-t+i|)(-1)dt + \int_{-1}^{0} (|-1-it|)(-i)dt \\
&= \int_{-1}^{1} |t|\, dt + i\int_{0}^{1} \sqrt{1+t^2}dt - \int_{-1}^{1} \sqrt{t^2+1}dt - i\int_{-1}^{0} \sqrt{1+t^2}dt
\end{aligned}$$

Using the change of variables $t = -u$ in the fourth integral, we can see that it is the same $i\int_{1}^{0} \sqrt{1+u^2}du$, which then cancels out with the second one. Hence:

$$\int_C |z|\, dz = \int_{-1}^{1} (|t| - \sqrt{1+t^2})dt = \int_{-1}^{0} (-t - \sqrt{1+t^2})dt + \int_{0}^{1} (t - \sqrt{1+t^2})dt$$

Again, changing variables $t = -u$, we can see that

$$\int_{-1}^{0} (-t - \sqrt{1+t^2})dt = \int_{0}^{1} (t - \sqrt{1+t^2})dt,$$

and so:

$$\int_C |z|\, dz = 2\int_{0}^{1} (t - \sqrt{t^2+1})dt = 2\int_{0}^{1} = 1 - \sqrt{2} - \ln\left(\sqrt{2} - 1\right).$$

(The antiderivative of $\sqrt{t^2+1}$ can be found in a table of integrals or computed using trigonometric subtitution and/or integration by parts.)

Exercise 3. Show that

$$\left| \int_{\{|z|=1\}} \frac{2z+1}{5+z^2} dz \right| \leq \frac{3\pi}{2}.$$

<u>**Solution:**</u> Let $|z| = 1$. Then:

$$|2z+1| \leq |2z| + |1| = 2|z| + 1 = 2(1) + 1 = 3.$$

For the denominator, we may parametrize the circle $\{|z| = 1\}$ as $z(t) = e^{it}, 0 \leq t \leq 2\pi$. Therefore $5 + z^2 = 5 + \left(e^{it}\right)^2 = 5 + e^{i2t} = (5 + \cos(2t)) + i\sin(2t)$ and so:

$$\begin{aligned}\left|5 + z^2\right| &= \sqrt{(5 + \cos(2t))^2 + \sin^2(2t)} \\ &= \sqrt{5^2 + 2\cos(2t) + \cos^2(2t) + \sin^2(2t)} \\ &= \sqrt{26 + 2\cos(2t)} \geq \sqrt{26 - 2} = \sqrt{24} > 4.\end{aligned}$$

Altogether, we obtain that:

$$\left|\frac{2z + 1}{5 + z^2}\right| \leq \frac{3}{4}.$$

Since the length of the paths is clearly 2π, the result follows.

Exercise 4. Find the maximum and minimum values of the function $f(z) = \left|z(1 - z)\right|$ on the disk $\{|z| \leq 1\}$.

Solution: We have $f(z) = \left|z(1 - z)\right| = |z||1 - z|$. Clearly, this function is the modulus of the function $z \mapsto z(1 - z)$ and so, since it never vanishes inside the disk $\{|z| \leq 1\}$, then by the maximum and minimum modulus principles, the maximum and minimum values are achieved on the boundary $\{|z| = 1\}$. On the boundary, we have $|z| = 1$ and so we may write $z = z(t) = e^{it}, 0 \leq t \leq 2\pi$ and

$$f(z) = |z||1 - z| = |1 - z| = \left|(1 - \cos(t)) + i\sin(t)\right| = \sqrt{(1 - \cos(t))^2 + \sin^2(t)}.$$

Simplifying further:

$$f(z) = \sqrt{1 - 2\cos(t) + \cos^2(t) + \sin^2(t)} = \sqrt{2 - 2\cos(t)} = \sqrt{2}\sqrt{1 - \cos(t)}.$$

Since $\cos(t)$ has a minimum value of -1 and a maximum value of 1, the maximum and minimum values of $f(z)$ are $\sqrt{2}\sqrt{2} = 2$ and 0, respectively.

Exercise 5. Let $f(x + iy) = u(x, y) + iv(x, y)$ be holomorphic on a region $A \subset \mathbb{C}$. Which of the following are also holomorphic?

- $f_1(x + iy) = u(x, y) - iv(x, y)$,
- $f_2(x + iy) = -u(x, y) - iv(x, y)$,
- $f_3(x + iy) = iu(x, y) - v(x, y)$.

Solution: Since f is holomorphic, then its real and imaginary parts satisfy the Cauchy-Riemann equations, meaning:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

and

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

- Let $u_1(x, y) = u(x, y)$ and $v_1(x, y) = -v(x, y)$. Then:

  $$\frac{\partial u_1}{\partial x} = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial(-v_1)}{\partial y} = -\frac{\partial v_1}{\partial y},$$

  and so the first Cauchy-Riemann equation is *not* satisfied for f_1.
  So f_1 is *not* holomorphic.

- Clearly, $f_2(x + iy) = -f(x + iy)$ and so it must be holomorphic.

- We can see that $f_3(x + iy) = i(u(x, y) + iv(x, y)) = if(x + iy)$ and so f_3 must be holomorphic.