Problem Set 2B: Practice Problem Set 2 – Solutions

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MAT 342 – Applied Complex Analysis
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NEVER DUE. Do the exercises for your own benefit. Practice makes perfect. On this note, keep in mind that the assignments are mostly for grading purposes and are thus not enough practice. If you have any questions, let me know.

Exercise 0. Review everything you've studied this week before proceeding!

Exercise 1. Suppose that f is analytic for $|z| < 2$ and let α be a complex constant. Show that:

$$\int_{\{|z|=1\}} \left(\mathrm{Re}(z) + \alpha\right) \frac{f(z)}{z} dz = \int_{\{|z|=1\}} \frac{1}{2} \left(\frac{1}{z^2} + \frac{2\alpha}{z}\right) f(z) dz.$$

Solution: Let us simplify the expression inside the integral. Let $z \in \{|z| = 1\}$. Then $\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \bar{z}$, and so:

$$\mathrm{Re}(z) = \frac{z + \bar{z}}{2} = \frac{z + 1/z}{2} = \frac{z^2 + 1}{2z}.$$

Therefore:

$$\begin{aligned}\int_{\{|z|=1\}} \left(\mathrm{Re}(z) + \alpha\right) \frac{f(z)}{z} dz &= \int_{\{|z|=1\}} \left(\frac{1}{2z}\left(z^2 + 1\right) \frac{f(z)}{z} + \alpha \frac{f(z)}{z}\right) dz \\ &= \frac{1}{2} \int_{\{|z|=1\}} f(z) dz + \int_{\{|z|=1\}} \frac{1}{2} \left(\frac{1}{z^2} + \frac{2\alpha}{z}\right) \frac{f(z)}{z^2} dz\end{aligned}$$

Since f is analytic on a domain containing the unit circle, the first integral is zero and that gives us the desired result.

Exercise 2. Suppose f is a continuous function on the unit circle which is bounded by some positive real M, i.e. $\left|f(e^{i\theta})\right| \leq M$. Suppose also that

$$\left|\int_{\{|z|=1\}} f(z) dz\right| = 2\pi M.$$

Show that $f(z) = c\bar{z}$ for some constant $c \in \mathbb{C}$ such that $|c| = M$.

Solution: We may rewrite the integral $\int_{\{|z|=1\}} f(z)dz$ as follows:

$$\int_{\{|z|=1\}} f(z)dz = \int_0^{2\pi} f\left(e^{i\theta}\right) ie^{i\theta} d\theta,$$

by parametrizing the unit circle the usual way.

We then have:

$$\begin{aligned} 2\pi M = \left|\int_{\{|z|=1\}} f(z)dz\right| &= \left|\int_0^{2\pi} f\left(e^{i\theta}\right) ie^{i\theta} d\theta\right| \\ &\leq \int_0^{2\pi} \left|f\left(e^{i\theta}\right) ie^{i\theta}\right| d\theta \\ &= \int_0^{2\pi} \left|f\left(e^{i\theta}\right)\right| d\theta \\ &\leq \int_0^{2\pi} M d\theta \\ &= 2\pi M. \end{aligned}$$

Therefore, we must have that:

$$\left|\int_0^{2\pi} f\left(e^{i\theta}\right) ie^{i\theta} d\theta\right| = \int_0^{2\pi} \left|f\left(e^{i\theta}\right) ie^{i\theta}\right| d\theta.$$

By squaring both sides and rewriting both absolute values in terms of real parts and imaginary parts, we can see that either $\operatorname{Re}\left(f\left(e^{i\theta}\right) ie^{i\theta}\right) = 0$ or $\operatorname{Im}\left(f\left(e^{i\theta}\right) ie^{i\theta}\right) = 0$, but not both. Without loss of generality assume that $\operatorname{Im}\left(f\left(e^{i\theta}\right) ie^{i\theta}\right) = 0$. The argument below can be replicated in the other case. Looking back at the inequality above, this implies that

$$\int_0^{2\pi} \left|\operatorname{Re}\left(f\left(e^{i\theta}\right) ie^{i\theta}\right)\right| d\theta = \left|\int_0^{2\pi} f\left(e^{i\theta}\right) ie^{i\theta} d\theta\right| = 2\pi M.$$

If $\left|\operatorname{Re}\left(f\left(e^{i\theta}\right) ie^{i\theta}\right)\right| < M$, we would have that

$$\int_0^{2\pi} \left|\operatorname{Re}\left(f\left(e^{i\theta}\right) ie^{i\theta}\right)\right| d\theta = \left|\int_0^{2\pi} f\left(e^{i\theta}\right) ie^{i\theta} d\theta\right| < 2\pi M,$$

contradicting the equality above. Therefore, since

$$\left|\operatorname{Re}\left(f\left(e^{i\theta}\right) ie^{i\theta}\right)\right| \leq \left|f\left(e^{i\theta}\right) ie^{i\theta}\right| = \left|f\left(e^{i\theta}\right)\right| \leq M,$$

it must be the case that $\left|\operatorname{Re}\left(f\left(e^{i\theta}\right) ie^{i\theta}\right)\right| = M$; i.e. $\operatorname{Re}\left(f\left(e^{i\theta}\right) ie^{i\theta}\right) = \pm M$.

Therefore, $f\left(e^{i\theta}\right) = \pm iMe^{-i\theta}$ and so, on the unit circle, $f(z) = c\bar{z}$ where $c = \pm iM$.

Exercise 3. Let $n, m \geq 1$ and $R > 1$. Show that:

$$\left| \int_{\{|z|=R\}} \frac{z^n}{z^m - 1} dz \right| \leq \frac{2\pi R^{n+1}}{R^m - 1}.$$

What can you say about the value of the integral as R gets very large (i.e. tends to ∞)?

Solution: Let z be in the circle $\{|z| = R\}$ so that $|z| = R$.

By the reverse triangle inequality, we have: $|z^m - 1| \geq \left| |z|^m - 1 \right| = |R^m - 1| = R^m - 1$ and so

$$\frac{1}{|z^m - 1|} \leq \frac{1}{R^m - 1}.$$

Therefore, for z in $\{|z| = R\}$, we have:

$$\left| \frac{z^n}{z^m - 1} \right| = \frac{|z|^n}{|z^m - 1|} = \frac{R^n}{|z^m - 1|} \leq \frac{R^n}{R^m - 1}.$$

Since the length of the circle $\{|z| = M\}$ is $2\pi R$, we obtain:

$$\left| \int_{\{|z|=R\}} \frac{z^n}{z^m - 1} dz \right| \leq (2\pi R) \left(\frac{R^n}{R^m - 1} \right) = \frac{2\pi R^{n+1}}{R^m - 1}.$$

Now:

$$\lim_{R \to \infty} \frac{2\pi R^{n+1}}{R^m - 1} = 2\pi \left(\lim_{R \to \infty} R^{n+1-m} \right).$$

If $n + 1 > m$, we cannot conclude anything because the inequality is then trivial. If $n + 1 < m$, we conclude that the limit of the integral as $|z| \to \infty$ vanishes. If $n + 1 = m$, we conclude that the limit of the integral as $|z| \to \infty$ is bounded by 2π.

Exercise 4. Let $f(z) = az^2 + b|z|^2 + c\bar{z}^2$ where $a, b, c \in \mathbb{C}$ are fixed. Using the limit-definition of holomorphicity, find the values of a, b and c for which f is holomorphic.

Solution: Let us rewrite $f(z)$ as $f(z) = az^2 + bz\bar{z} + c\bar{z}^2$ since $|z|^2 = z\bar{z}$. Let z_0 be arbitrary. Then:

$$\begin{aligned}
\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} &= \lim_{z \to z_0} \frac{a(z^2 - z_0^2) + b(z\bar{z} - z_0\bar{z}_0) + c(\bar{z}^2 - \bar{z}_0^2)}{z - z_0} \\
&= \lim_{z \to z_0} \frac{a(z + z_0)(z - z_0)}{z - z_0} + \frac{bz\bar{z} - bz\bar{z}_0 + bz\bar{z}_0 - bz_0\bar{z}_0}{z - z_0} + \frac{c(\bar{z} + \bar{z}_0)(\bar{z} - \bar{z}_0)}{z - z_0} \\
&= a \left(\lim_{z \to z_0} z + z_0 \right) + bz \left(\lim_{z \to z_0} \frac{\overline{z - z_0}}{z - z_0} \right) + b\bar{z}_0 \left(\lim_{z \to z_0} \frac{z - z_0}{z - z_0} \right) + 2c\bar{z}_0 \left(\lim_{z \to z_0} \frac{\overline{z - z_0}}{z - z_0} \right) \\
&= 2az_0 + b\bar{z}_0 + (bz + 2c\bar{z}_0) \left(\lim_{z \to z_0} \frac{\overline{z - z_0}}{z - z_0} \right).
\end{aligned}$$

Since the limit $\lim_{z \to z_0} \dfrac{\overline{z - z_0}}{z - z_0}$, the limit quotient exists if and only if $bz + 2c\bar{z}_0 = 0$ and so for f to be holomorphic, we must have that $b = c = 0$.

Exercise 5. Let p be the polynomial defined by $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$. Show that for any radius $R > 0$ and $z_0 \in \mathbb{C}$:

$$p(z_0) = \frac{1}{2\pi i} \int_{\{|z-z_0|=R\}} \frac{p(z)}{z - z_0} dz.$$

Solution: Let us examine the integral $\int_{\{|z-z_0|=R\}} \frac{p(z)}{z-z_0} dz$:

$$\begin{aligned}
\int_{\{|z-z_0|=R\}} \frac{p(z)}{z-z_0} dz &= \int_{\{|z-z_0|=R\}} \frac{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 + a_0}{z - z_0} dz \\
&= a_n \left(\int_{\{|z-z_0|=R\}} \frac{z^n}{z-z_0} dz \right) + \cdots + a_0 \left(\int_{\{|z-z_0|=R\}} \frac{1}{z-z_0} dz \right) \\
&= a_n \left(\int_{\{|z-z_0|=R\}} \left(\frac{z^n - z_0^n}{z - z_0} + \frac{z_0^n}{z-z_0} \right) dz \right) + \cdots + a_0 \left(\int_{\{|z-z_0|=R\}} \frac{1}{z-z_0} dz \right).
\end{aligned}$$

Now let $1 \leq k \leq n$. Then:

$$\frac{z^k - z_0^k}{z - z_0} = z^{k-1} + z^{k-2} z_0 + \cdots + z z_0^{k-2} + z_0^{k-1}.$$

This comes from the factorization of $a^n - b^n$ which holds for complex numbers. You may also prove this by replacing z by z/z_0 in the identity

$$\frac{z^k - 1}{z - 1} = 1 + z + z^2 + \cdots + z^{k-1}.$$

The expression at the right is a polynomial of degree $k-1$ whose coefficients are powers of z_0. Therefore, it is holomorphic and so by Cauchy's theorem:

$$\int_{\{|z-z_0|=R\}} \frac{z^k - z_0^k}{z - z_0} dz = 0,$$

where $1 \leq k \leq n$. Therefore:

$$\begin{aligned}
\int_{\{|z-z_0|=R\}} \frac{p(z)}{z-z_0} dz &= a_n \left(\int_{\{|z-z_0|=R\}} \frac{z_0^n}{z-z_0} dz \right) + \cdots + a_0 \left(\int_{\{|z-z_0|=R\}} \frac{1}{z-z_0} dz \right) \\
&= \left(a_n z_0^n + a_{n-1} z_0^{n-1} + \cdots + a_0 \right) \left(\int_{\{|z-z_0|=R\}} \frac{1}{z-z_0} dz \right) \\
&= p(z_0)(2\pi i),
\end{aligned}$$

using the fact that $\int_{\{|z-z_0|=R\}} \frac{1}{z-z_0} dz = 2\pi i$; which proves the result.