Problem Set 2A: Assignment 2 – Solutions

Instructor: El Mehdi Ainasse
MAT 342 – Applied Complex Analysis
Summer Session II 2019

DUE: July 25th, 2019 – AT THE BEGINNING OF CLASS.

Exercise 0. Review everything you've studied this week before proceeding!

Exercise 1. Compute the complex log of each of the numbers $2, i, 1+i$ and $\frac{1+i\sqrt{3}}{2}$.

Solution:

- $\log(2) = \ln(|2|) + i\arg(2) + i2\pi n = \ln(2) + i2\pi n, n \in \mathbb{Z}$.
- $\log(i) = \ln\left(|i|\right) + i\arg(i) + i2\pi n = i\pi/2 + i2\pi n, n \in \mathbb{Z}$.
- $\log(1+i) = \ln(|1+i|) + i\arg(1+i) + i2\pi n = \ln(\sqrt{2}) + i\pi/4 + i2\pi n, n \in \mathbb{Z}$.
- $\log\left(\frac{1+i\sqrt{3}}{2}\right) = \ln\left(\left|\frac{1+i\sqrt{3}}{2}\right|\right) + i\arg\left(\frac{1+i\sqrt{3}}{2}\right) + i2\pi n = i\pi/3 + i2\pi n, n \in \mathbb{Z}$.

Exercise 2.

1. Using the geometric series, show that for $z \neq 1$:

$$1 + 2z + 3z^2 + \cdots + nz^{n-1} = \frac{1-z^n}{(1-z)^2} - \frac{nz^n}{1-z}.$$

2. Show that $e^{\bar{z}} = \overline{e^z}$.

Solution:

1. Recall the identity $1 + z + z^2 + \cdots + z^n = \frac{z^{n+1}-1}{z-1} = \frac{1-z^{n+1}}{1-z}$ for any integer $n \geq 1$. Differentiating both sides:

$$\begin{aligned} 0 + 1 + 2z + \cdots + nz^{n-1} &= \frac{(-(n+1)z^n)(1-z) - (-1)(1-z^{n+1})}{(1-z)^2} \\ &= \frac{-(n+1)z^n + (n+1)z^{n+1} + 1 - z^{n+1}}{(1-z)^2} \\ &= \frac{(1-z^n) + (1-z)nz^n}{(1-z)^2} \\ &= \frac{1-z^n}{(1-z)^2} - \frac{nz^n}{1-z}. \end{aligned}$$

2. Let $z = x + iy$. Then:

$$\begin{aligned} e^{\bar{z}} = e^{x-iy} = e^x(\cos(-y) + i\sin(-y)) &= e^x\cos(y) - ie^x\sin(y) \\ &= \overline{e^x\cos(y) - ie^x\sin(y)} \\ &= e^x\cos(y) + ie^x\sin(y) \\ &= e^x(\cos(y) + i\sin(y)) \\ &= e^z \end{aligned}$$

Exercise 3. Let f be holomorphic on A and let $A^* = \{z \in \mathbb{C} : \bar{z} \in A\}$ be the set of points $\mathbb{C}$ whose conjugates are in A. Suppose f is holomorphic on A.

1. Show that the function g defined by $g(z) = \overline{f(\bar{z})}$ is holomorphic on A^*.
2. Show that $g'(z) = \overline{f'(\bar{z})}$.

Solution:

1. Let $z_0 \in A^*$ be arbitrary. Then $\bar{z}_0 \in A$ and so f is holomorphic at $\bar{z}_0$. Let z approach z_0 with $z \in A^*$ so that $\bar{z} \in A$. Then the limit

$$\lim_{\bar{z}\to\bar{z}_0} \frac{f(\bar{z}) - f(\bar{z}_0)}{\bar{z} - \bar{z}_0}$$

exists. But upon closer examination:

$$\lim_{\bar{z}\to\bar{z}_0} \frac{f(\bar{z}) - f(\bar{z}_0)}{\bar{z} - \bar{z}_0} = \lim_{\bar{z}\to\bar{z}_0} \frac{f(\bar{z}) - f(\bar{z}_0)}{\overline{z - z_0}} = \lim_{\bar{z}\to\bar{z}_0} \overline{\left(\frac{\overline{f(\bar{z})} - \overline{f(\bar{z}_0)}}{z - z_0}\right)} = \lim_{\bar{z}\to\bar{z}_0} \overline{\left(\frac{g(z) - g(z_0)}{z - z_0}\right)}.$$

But $z \to z_0$ is equivalent to $\bar{z} \to \bar{z}_0$ and so:

$$\lim_{\bar{z}\to\bar{z}_0} \frac{f(\bar{z}) - f(\bar{z}_0)}{\bar{z} - \bar{z}_0} = \lim_{\bar{z}\to\bar{z}_0} \overline{\left(\frac{g(z) - g(z_0)}{z - z_0}\right)}.$$

Therefore, the limit $\lim_{z\to z_0} \dfrac{g(z) - g(z_0)}{z - z_0}$ exists for any $z_0 \in A^*$ and so g is holomorphic on A^* by definition.

2. Let $z_0 \in A^*$ be arbitrary. By the computation above:

$$g'(z_0) = \lim_{z\to z_0} \frac{g(z) - g(z_0)}{z - z_0} = \lim_{\bar{z}\to\bar{z}_0} \overline{\left(\frac{f(\bar{z}) - f(\bar{z}_0)}{\bar{z} - \bar{z}_0}\right)} = \overline{f'(\bar{z}_0)},$$

and since $z_0 \in A^*$ was arbitrary, the result follows.

Exercise 4. Compute the length of the contour defined by $\gamma(t) = e^t\left(\cos(t) + i\sin(t)\right)$ for $-\pi \leq t \leq \pi$.

<u>Solution:</u> We have $\gamma'(t) = e^t(-\sin(t) + i\cos(t)) + e^t(\cos(t) + i\sin(t))$ by the product rule and so

$$\gamma'(t) = e^t\left((\cos(t) - \sin(t)) + i(\cos(t) + \sin' t))\right),$$

whence

$$\begin{aligned}\left|\gamma'(t)\right|^2 &= e^{2t}\left((\cos^2(t) + \sin^2(t) - 2\cos(t)\sin(t)) + (\cos^2(t) + 2\sin(t)\cos(t) + \sin^2(t))\right) \\ &= e^{2t}\left(2(\cos^2(t) + \sin^2(t)\right) = 2e^{2t}.\end{aligned}$$

Therefore, the length L_γ of the contour is:

$$L_\gamma = \int_{-4}^{4}\left|\gamma'(t)\right|dt = \int_{-4}^{4}\sqrt{2e^{2t}}dt = \int_{-4}^{4}\sqrt{2}e^t dt = \sqrt{2}\left[e^t\right]_{t=-4}^{t=4} = \sqrt{2}\left(e^4 + e^{-4}\right).$$

Exercise 5. Compute the integral

$$\int_{\{|z|=1\}}\left(\frac{z\sin(z)}{z+2} + \bar{z}\right)dz.$$

<u>Solution:</u>

$$\int_{\{|z|=1\}}\left(\frac{z\sin(z)}{z+2} + \bar{z}\right)dz = \int_{\{|z|=1\}}\frac{z\sin(z)}{z+2}dz + \int_{\{|z|=1\}}\bar{z}dz.$$

The only point of discontinuity for the function $z \mapsto \dfrac{z\sin(z)}{z+2}$ is $z = -2$ which lies outside the circle and the region which it bounds: the unit disk. The numerator and the denominator define functions which are holomorphic on the unit disk up to its boundary. Furthermore, the numerator never vanishes inside the unit disk nor does it vanish on on its boundary (the circle). Therefore, the function $z \mapsto \dfrac{z\sin(z)}{z+2}$ is holomorphic on the unit disk (up to the boundary) and so by Cauchy's theorem, the first integral vanishes.

The complex conjugate function is not holomorphic anywhere and so the second integral needs to be explicitly computed. Let us parametrize the unit circle by $z(t) = e^{it}, 0 \leq t \leq 2\pi$. Then:

$$\int_{\{|z|=1\}}\bar{z}dz = \int_0^{2\pi}\overline{e^{it}}(ie^{it})dt = i\int_0^{2\pi}e^{-it}e^{it}dt = i\int_0^{2\pi}dt = 2\pi i.$$

All in all:

$$\int_{\{|z|=1\}}\left(\frac{z\sin(z)}{z+2} + \bar{z}\right)dz = 2\pi i.$$