# Problem Set 1B: Practice Problem Set 1 – Solutions

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MAT 342 – Applied Complex Analysis
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**NEVER DUE. Do the exercises for your own benefit. Practice makes perfect. On this note, keep in mind that the assignments are mostly for grading purposes and are thus not enough practice. If you have any questions, let me know.**

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**Exercise 0.** Review everything you've studied this week before proceeding!

**Exercise 1.** Let $\alpha \in \mathbb{C}$ such that $|\alpha| < 1$. Prove that

$$\left|\frac{z-\alpha}{1-\bar{\alpha}z}\right| = 1 \text{ if and only if } |z| = 1.$$

**Solution:**

$$\begin{aligned}
\left|\frac{z-\alpha}{1-\bar{\alpha}z}\right| = 1 &\iff |z-\alpha| = |1-\bar{\alpha}z| \\
&\iff |z-\alpha|^2 = |1-\bar{\alpha}z|^2 \\
&\iff (z-\alpha)(\overline{z-\alpha}) = (1-\bar{\alpha}z)(\overline{1-\bar{\alpha}z}) \\
&\iff (z-\alpha)(\bar{z}-\bar{\alpha}) = (1-\bar{\alpha}z)(1-\alpha\bar{z}) \\
&\iff z\bar{z} - \alpha\bar{z} - z\bar{\alpha} + \alpha\bar{\alpha} = 1 - \bar{\alpha}z - \alpha\bar{z} + \bar{\alpha}z\alpha\bar{z} \\
&\iff z\bar{z} + \alpha\bar{\alpha} = 1 + z\bar{z}\alpha\bar{\alpha} \\
&\iff z\bar{z} - 1 = z\bar{z}\alpha\bar{\alpha} - \alpha\bar{\alpha} \\
&\iff |z|^2 - 1 = |z|^2|\alpha|^2 - |\alpha|^2 \\
&\iff |z|^2 - 1 = |\alpha|^2\left(|z|^2 - 1\right) \\
&\iff \left(1 - |\alpha|^2\right)\left(|z|^2 - 1\right) = 0
\end{aligned}$$

But since $|\alpha| < 1$, it is also the case that $|\alpha|^2 < 1$. Therefore, the latter happens if and only if $|z|^2 - 1 = 0$; which happens if and only $|z| = 1$. This completes the proof.

**Exercise 2.** Suppose that $z \in \mathbb{C}$ and $z \neq 0$. Show that $z + \dfrac{1}{z}$ is a real number if and only if $\mathrm{Im}(z) = 0$ or $|z| = 1$.

**<u>Solution:</u>** Suppose that $z \neq 0$.

$$
\begin{aligned}
z + \frac{1}{z} \in \mathbb{R} &\iff z + \frac{1}{z} = \overline{z + \frac{1}{z}} \\
&\iff z - \bar{z} = \overline{\left(\frac{1}{z}\right)} - \frac{1}{z} \\
&\iff z - \bar{z} = \frac{1}{\bar{z}} - \frac{1}{z} \\
&\iff z - \bar{z} = \frac{z - \bar{z}}{\bar{z}z} \\
&\iff z - \bar{z} - \frac{z - \bar{z}}{z\bar{z}} = 0 \\
&\iff z - \bar{z} - \frac{z - \bar{z}}{|z|^2} = 0 \\
&\iff (z - \bar{z})\left(1 - \frac{1}{|z|^2}\right) = 0 \\
&\iff z - \bar{z} = 0 \,\text{or}\, 1 - \frac{1}{|z|^2} = 0 \\
&\iff z = \bar{z} \,\text{or}\, 1 = \frac{1}{|z|^2} \\
&\iff z \in \mathbb{R} \text{ or } 1 = |z|^2 \\
&\iff \mathrm{Im}(z) = 0 \text{ or } |z| = 1
\end{aligned}
$$

You may also want to write $z$ as $z = x + iy$ and try to prove that $y = 0$ or $x^2 + y^2 = 1$.

**Exercise 3.** Show that for any complex number $z \in \mathbb{C}$:

$$|z| \leq |\mathrm{Re}(z)| + |\mathrm{Im}(z)| \,.$$

For which points on the complex plane do we have that

$$|z| = |\mathrm{Re}(z)| + |\mathrm{Im}(z)|?$$

(When do we have equality instead of an inequality?)

**<u>Solution:</u>** Let us write $z = x + iy$ where $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$.
We need to show that:

$$\sqrt{x^2 + y^2} \leq |x| + |y| \,,$$

which is equivalent to:

$$x^2 + y^2 \leq \big(|x| + |y|\big)^2 = |x|^2 + 2|x||y| + |y|^2 = x^2 + y^2 + 2|xy| \,.$$

But this is clearly true by subtracting $x^2 + y^2$ from both sides.
Now as we have seen, $|z| \leq \big|\text{Re}(z)\big| + \big|\text{Im}(z)\big|$
is equivalent to

$$x^2 + y^2 \leq x^2 + y^2 + 2|xy|\,.$$

So if $2|xy| = 0$, we have equality. Therefore, if $x = 0$ or $y = 0$, we have equality. In other words, equality happens for complex numbers which are either purely real or purely imaginary. Or yet again, for points in the complex plane which are either on the real axis or the imaginary axis.

**Exercise 4.** Let $n \geq 1$ be an integer.

1. Prove that for any complex number $z \neq 1$:

$$1 + z + \cdots + z^n = \frac{z^n - 1}{z - 1},$$

**without using induction**.

2. Show that:

$$1 + \cos(\theta) + \cos(2\theta) + \cdots + \cos(n\theta) = \frac{1}{2} + \frac{\sin\left(\left(n + \frac{1}{2}\right)\theta\right)}{2\sin\left(\frac{\theta}{2}\right)}.$$

**Solution:**

1. Let $S = 1 + z + z^2 + \cdots + z^n$ where $z \neq 1, z \in \mathbb{C}$. By multiplying $S$ by $z$, we have:

$$\begin{aligned} zS &= z + z^2 + \cdots + z^n + z^{n+1} \\ &= 1 - 1 + z + z^2 + \cdots + z^n + z^{n+1} \\ &= (1 + z + \cdots + z^n) + (z^{n+1} - 1) = S + (z^{n+1} - 1), \end{aligned}$$

and so:

$$zS - S = z^{n+1} - 1,$$

which gives us the result by factoring by $S$ in the left-side term and then dividing through by $z - 1$.

2. Recall that:

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

With this in mind, it would be judicious to apply the identity from 1 for $z = e^{i\theta}$ and $z = e^{-i\theta}$, assuming of course $z \neq 1$. In those cases, we obtain:

$$1 + e^{i\theta} + e^{i2\theta} + \cdots + e^{in\theta} = \frac{e^{i(n+1)\theta} - 1}{e^{i\theta} - 1},$$

$$1 + e^{-i\theta} + e^{-i2\theta} + \cdots + e^{-in\theta} = \frac{e^{-i(n+1)\theta} - 1}{e^{-i\theta} - 1}.$$

Adding up the two equalities term by term and dividing all through by 2, we have:

$$1 + \frac{e^{i\theta} + e^{-i\theta}}{2} + \frac{e^{i2\theta} + e^{-i2\theta}}{2} + \cdots + \frac{e^{in\theta} + e^{-in\theta}}{2} = \frac{1}{2}\left(\frac{e^{i(n+1)\theta} - 1}{e^{i\theta} - 1} + \frac{e^{-i(n+1)\theta} - 1}{e^{-i\theta} - 1}\right).$$

Clearly, the quantity on the left is

$$1 + \cos(\theta) + \cos(2\theta) + \cdots + \cos(n\theta).$$

The quantity on the right needs more simplification. Recall also that

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

$$\begin{aligned}
\frac{1}{2}\left(\frac{e^{i(n+1)\theta} - 1}{e^{i\theta} - 1} + \frac{e^{-i(n+1)\theta} - 1}{e^{-i\theta} - 1}\right) &= \frac{1}{2}\left(\frac{e^{i(n+1)\theta} - 1}{e^{i\theta/2}\big(e^{i\theta/2} - e^{-i\theta/2}\big)} + \frac{e^{-i(n+1)\theta} - 1}{e^{-i\theta/2}\big(e^{-i\theta/2} - e^{i\theta/2}\big)}\right) \\
&= \frac{1}{2}\left(\frac{(e^{i(n+1)\theta} - 1)e^{-i\theta/2}}{e^{i\theta/2} - e^{-i\theta/2}} + \frac{(e^{-i(n+1)\theta} - 1)e^{i\theta/2}}{e^{-i\theta/2} - e^{i\theta/2}}\right) \\
&= \frac{1}{2}\left(\frac{e^{i(n+1/2)\theta} - e^{-i\theta/2}}{e^{i\theta/2} - e^{-i\theta/2}} - \frac{e^{-i(n+1/2)\theta} - e^{i\theta/2}}{e^{i\theta/2} - e^{-i\theta/2}}\right) \\
&= \frac{1}{2}\left(\frac{e^{i(n+1/2)\theta} - e^{-i(n+1/2)\theta} + e^{i\theta/2} - e^{-i\theta/2}}{e^{i\theta/2} - e^{-i\theta/2}}\right) \\
&= \frac{1}{2}\left(\frac{2i\sin\Big(\big(n + 1/2\big)\,\theta\Big) + 2i\sin(\theta/2)}{2i\sin(\theta/2)}\right) \\
&= \frac{1}{2} + \frac{\sin\left(\left(n + \frac{1}{2}\right)\theta\right)}{2\sin\left(\frac{\theta}{2}\right)}.
\end{aligned}$$

Therefore, the identity which was to be shown holds indeed.

**Exercise 5.** Use the $(\varepsilon, \delta)$-definition of continuity, prove that the functions defined below are continuous.

1. $f(z) = \bar{z}$.
2. $g(z) = \mathrm{Re}(z)$.

3. $h(z) = \mathrm{Im}(z)$.
   (a) Prove this without using the fact that $g(z) = \mathrm{Re}(z)$ is continuous.
   (b) Now prove this using the fact that $g(z) = \mathrm{Re}(z)$ is continuous.
4. $k(z) = |z|$.

**Solution:** In what follows, let $\varepsilon > 0$ and $z_0$ be arbitrary. Let $\delta > 0$ be a positive number, to be found, such that $|z - z_0| < \delta$.

1. Let $f(z) = \bar{z}$. We have:
$$\left|f(z) - f(z_0)\right| = |\bar{z} - \bar{z_0}| = \left|\overline{z - z_0}\right| = |z - z_0| < \delta.$$
Therefore, choosing $\delta = \varepsilon$ establishes continuity.

2. Let $g(z) = \mathrm{Re}(z)$. We know that
$$\left|\mathrm{Re}(z)\right| \leq |z|,$$
and so:
$$\left|g(z) - g(z_0)\right| = \left|\mathrm{Re}(z) - \mathrm{Re}(z_0)\right| = \left|\mathrm{Re}(z - z_0)\right| \leq |z - z_0| < \delta.$$
Therefore, choosing $\delta = \varepsilon$ establishes continuity.

3. Let $h(z) = \mathrm{Im}(z)$.
   (a) We know that
$$\left|\mathrm{Im}(z)\right| \leq |z|,$$
   and so:
$$\left|h(z) - h(z_0)\right| = \left|\mathrm{Im}(z) - \mathrm{Im}(z_0)\right| = \left|\mathrm{Im}(z - z_0)\right| \leq |z - z_0| < \delta.$$
   Therefore, choosing $\delta = \varepsilon$ establishes continuity.
   (b) Since $g(z) = \mathrm{Re}(z)$ is continuous, then for any $\tilde{\varepsilon} > 0$, there is a $\tilde{\delta} > 0$ such that if $|z - z_0| < \tilde{\delta}$ then $\left|g(z) - g(z_0)\right| < \tilde{\varepsilon}$. We know that $z = \mathrm{Re}(z) + i\mathrm{Im}(z)$. Therefore:
$$\mathrm{Im}(z) = \frac{1}{i}(z - \mathrm{Re}(z)) = -i(z - \mathrm{Re}(z)).$$
   Using the triangle inequality, we have:
$$\begin{aligned}\left|h(z) - h(z_0)\right| &= \left|\mathrm{Im}(z) - \mathrm{Im}(z_0)\right| \\ &= \left|\mathrm{Im}(z - z_0)\right| \\ &= \left|-i((z - z_0) - \mathrm{Re}(z - z_0))\right| \\ &= |-i|\left|(z - z_0) - \mathrm{Re}(z - z_0)\right| \\ &= \left|(z - z_0) - \mathrm{Re}(z - z_0)\right| \\ &\leq |z - z_0| + \left|\mathrm{Re}(z - z_0)\right| \\ &= |z - z_0| + \left|g(z) - g(z_0)\right| \\ &\leq \tilde{\delta} + \tilde{\varepsilon}.\end{aligned}$$
   Therefore, choosing $\delta = \tilde{\delta}$ and $\tilde{\varepsilon} = \varepsilon - \tilde{\delta}$ establishes continuity.

4. Let $k(z) = |z|$. Then by the reverse triangle inequality:

$$\big|k(z) - k(z_0)\big| = \big||z| - |z_0|\big| \leq |z - z_0| \leq \delta.$$

Therefore, choosing $\delta = \varepsilon$ establishes continuity.