Problem Set 1B: Practice Problem Set 1

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MAT 342 – Applied Complex Analysis
Summer Session II 2019

NEVER DUE. Do the exercises for your own benefit. Practice makes perfect. On this note, keep in mind that the assignments are mostly for grading purposes and are thus not enough practice. If you have any questions, let me know.

Exercise 0. Review everything you've studied this week before proceeding!

Exercise 1. Let $\alpha \in \mathbb{C}$ such that $|\alpha| < 1$. Prove that

$$\left|\frac{z-\alpha}{1-\bar{\alpha}z}\right| = 1 \text{ if and only if } |z| = 1.$$

Exercise 2. Suppose that $z \in \mathbb{C}$ and $z \neq 0$. Show that $z + \dfrac{1}{z}$ is a real number if and only if $\mathrm{Im}(z) = 0$ or $|z| = 1$.

Exercise 3. Show that for any complex number $z \in \mathbb{C}$:

$$|z| \leq \left|\mathrm{Re}(z)\right| + \left|\mathrm{Im}(z)\right|.$$

For which points on the complex plane do we have that

$$|z| = \left|\mathrm{Re}(z)\right| + \left|\mathrm{Im}(z)\right|?$$

(When do we have equality instead of an inequality?)

Exercise 4. Let $n \geq 1$ be an integer.

1. Prove that for any complex number $z \neq 1$:

$$1 + z + \cdots + z^n = \frac{z^{n+1} - 1}{z - 1},$$

without using induction.

2. Show that:

$$1 + \cos(\theta) + \cos(2\theta) + \cdots + \cos(n\theta) = \frac{1}{2} + \frac{\sin\left(\left(n + \frac{1}{2}\right)\theta\right)}{2\sin\left(\frac{\theta}{2}\right)}.$$

Exercise 5. Use the (ε, δ)-definition of continuity, prove that the functions defined below are continuous.

1. $f(z) = \bar{z}$.
2. $g(z) = \mathrm{Re}(z)$.
3. $h(z) = \mathrm{Im}(z)$.
   (a) Prove this without using the fact that $g(z) = \mathrm{Re}(z)$ is continuous.
   (b) Now prove this using the fact that $g(z) = \mathrm{Re}(z)$ is continuous.
4. $k(z) = |z|$.