Problem Set 1A: Assignment 1 – Solutions

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Exercise 0. Review everything you've studied this week before proceeding!

Exercise 1. Express the following in the form $x + iy$.

(a) $\dfrac{a+ib}{a-ib} - \dfrac{a-ib}{a+ib}$. ($a$ and b are real numbers.)

(b) $\dfrac{\sqrt{1+a^2}+ia}{a-i\sqrt{1+a^2}}$. ($a$ is a real number.)

Solution:

(a)

$$\begin{aligned}\frac{a+ib}{a-ib} - \frac{a-ib}{a+ib} &= \frac{(a+ib)(a+ib)-(a-ib)(a-ib)}{(a-ib)(a+ib)} \\ &= \frac{(a^2+2iab+i^2b^2)-(a^2-2iab+i^2b^2)}{a^2+b^2} \\ &= \frac{a^2+2iab-b^2-a^2+2iab+b^2}{a^2+b^2} \\ &= \frac{4iab}{a^2+b^2} = 0 + i\left(\frac{4ab}{a^2+b^2}\right)\end{aligned}$$

(b)

$$\begin{aligned}\frac{\sqrt{1+a^2}+ia}{a-i\sqrt{1+a^2}} &= \frac{(\sqrt{1+a^2}+ia)(a+i\sqrt{1+a^2})}{(a-i\sqrt{1+a^2})(a+i\sqrt{1+a^2})} \\ &= \frac{a\sqrt{1+a^2}+i(1+a^2)+ia^2-a\sqrt{1+a^2}}{a^2+(\sqrt{1+a^2})^2} \\ &= \frac{i(1+2a^2)}{2a^2+1} = i = 0 + 1i\end{aligned}$$

Exercise 2. Let $z_1 = x_1 + iay_1$ and $z_2 = x_2 - i\dfrac{b}{y_1}$ where x_1, y_1, x_2, a and b are real numbers and $y_1 \neq 0$. Determine a condition on y_1 so that $z_1^{-1} + z_2^{-1}$ is real.

__**Solution:**__ We have:

$$\begin{aligned} z_1^{-1} + z_2^{-1} &= \frac{\overline{z_1}}{|z_1|^2} + \frac{\overline{z_2}}{|z_2|^2} \\ &= \frac{x_1 - iay_1}{x_1^2 + a^2y_1^2} + \frac{x_2 + i\dfrac{b}{y_1}}{x_2^2 + \dfrac{b^2}{y_1^2}} \\ &= \frac{x_1}{x_2^2 + a^2y_1^2} + \frac{x_2}{x_2^2 + \dfrac{b^2}{y_1^2}} + i\left(-\frac{ay_1}{x_1^2 + a^2y_1^2} + \frac{\dfrac{b}{y_1}}{x_2^2 + \dfrac{b^2}{y_1^2}}\right). \end{aligned}$$

For $z_1^{-1} + z_2^{-1}$ to be real, we need the imaginary part to be zero; i.e.:

$$-\frac{ay_1}{x_1^2 + a^2y_1^2} + \frac{\dfrac{b}{y_1}}{x_2^2 + \dfrac{b^2}{y_1^2}} = 0.$$

$$\begin{aligned} -\frac{ay_1}{x_1^2 + a^2y_1^2} + \frac{\dfrac{b}{y_1}}{x_2^2 + \dfrac{b^2}{y_1^2}} = 0 &\iff \frac{\dfrac{b}{y_1}}{x_2^2 + \dfrac{b^2}{y_1^2}} = \frac{ay_1}{x_1^2 + a^2y_1^2} \\ &\iff \frac{b}{y_1}(x_1^2 + a^2y_1^2) = ay_1\left(x_2^2 + \frac{b^2}{y_1^2}\right) \\ &\iff bx_1^2 + ba^2y_1^2 = ay_1^2\left(x_2^2 + \frac{b^2}{y_1^2}\right) \\ &\iff bx_1^2 + ba^2y_1^2 = ax_2^2y_1^2 + ab^2 \\ &\iff ba^2y_1^2 - ax_2^2y_1^2 = ab^2 - bx_1^2 \\ &\iff (ba^2 - ax_2^2)y_1^2 = ab^2 - bx_1^2 \\ &\iff y_1^2 = \frac{ab^2 - bx_1^2}{ba^2 - ax_2^2}, \end{aligned}$$

and that is the condition on y_1.

Exercise 3. Compute the possible square roots of $\dfrac{1 - i\sqrt{3}}{2}$ and $1 + i\sqrt{3}$.

Solution:

- Let us write $(a+ib)^2 = \frac{1-i\sqrt{3}}{2} = \frac{1}{2} - i\frac{\sqrt{3}}{2}$. Then:

$$a^2 - b^2 + 2iab = \frac{1}{2} - i\frac{\sqrt{3}}{2},$$

  and so $a^2 - b^2 = \frac{1}{2}$ and $2ab = -\frac{\sqrt{3}}{2}$. So then $2a^2 = 2b^2 + 1$ and $4a^2b^2 = \frac{3}{4}$. Dividing the second equation by 2 and subtituting $2b^2+1$ for $2a^2$, we obtain:

$$\begin{aligned}(2b^2+1)(2b^2) = \frac{3}{4} &\iff 4b^4 + 2b^2 = \frac{3}{4} \\ &\iff 16b^4 + 8b^2 - 3 = 0.\end{aligned}$$

  Letting $t = b^2 > 0$, this gives us the quadratic equation $16t^2 + 8t - 3 = 0$ whose solutions are $-3/4$ and $1/4$. But since t is positive, the only solution is $1/4$ and so $b^2 = 1/4$ so that $b = \pm 1/2$. This also tells us that $2a^2 = 2(1/4) + 1 = 3/2$ and so $a = \pm\sqrt{3}/2$. But since the product of a and b is negative, they must have opposite signs. Therefore, the possible square roots of $\frac{1-i\sqrt{3}}{2}$ are $\frac{\sqrt{3}}{2} - \frac{1}{2}i$ and $-\frac{\sqrt{3}}{2} + \frac{1}{2}i$.

- Let us write $(a+ib)^2 = 1 + i\sqrt{3}$. Then:

$$a^2 - b^2 + 2iab = 1 + i\sqrt{3},$$

  and so $a^2 - b^2 = 1$ and $2ab = \sqrt{3}$. So then $a^2 = b^2 + 1$ and $4a^2b^2 = 3$. Dividing the second equation by 4 and subtituting $b^2 + 1$ for a^2, we obtain:

$$4(b^2+1)b^2 = 3 \iff 4b^4 + 4b^2 - 3 = 0.$$

  Letting $t = b^2 > 0$, this gives us the quadratic equation $4t^2 + 4t - 3 = 0$ whose solutions are $t = -3/2$ and $t = 1/2$. Since $t > 0$, the only solution is $1/2$ and so $b^2 = 1/2$ meaning that $b = \pm 1/\sqrt{2}$. So then $a^2 = 1/2 + 1 = 3/2$ so that $a = \pm\sqrt{3}/\sqrt{2}$. However, as the product of a and b is negative, they will have opposite signs. Therefore, the possible square roots of $1 + i\sqrt{3}$ are $-\frac{\sqrt{3}}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ and $\frac{\sqrt{3}}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$.

Exercise 4. Find all the possible solutions of $z^{1-i} = 4$.

Solution:

$$\begin{aligned}z^{1-i} = 4 &\iff \left(e^{\log(z)}\right)^{1-i} = e^{\ln(4)+2k\pi i}, k \in \mathbb{Z} \\ &\iff (1-i)\log(z) = 2\ln(2) + 2k\pi i, k \in \mathbb{Z} \\ &\iff \log(z) = (\ln(2) - k\pi) + i(\ln(2) + k\pi), k \in Z.\end{aligned}$$

In the last step, we divided by $1-i$. By the definition of $\log(z)$:

$$z = e^{(\ln(2)-k\pi)+i(\ln(2)+k\pi)}, k \in \mathbb{Z}.$$

Upon simplification, the solutions are

$$z = 2e^{-k\pi}e^{i(\ln(2)+k\pi)}, k \in \mathbb{Z}.$$

Exercise 5. Find all the possible values of $\sin^{-1}(1/2)$ using the complex sine function.

Solution: Let $\theta = \sin^{-1}(1/2)$.

$$\begin{aligned}
\theta = \sin^{-1}(1/2) &\iff \sin(\theta) = 1/2 \\
&\iff \frac{e^{i\theta} - e^{-i\theta}}{2i} = 1/2 \\
&\iff e^{i\theta} - e^{-i\theta} = i \\
&\iff \left(e^{i\theta}\right)^2 - 1 = ie^{i\theta} \\
&\iff \left(e^{i\theta}\right)^2 - ie^{i\theta} - 1 = 0.
\end{aligned}$$

This last equation is a complex quadratic equation in $e^{i\theta}$. Let $t = e^{i\theta}$. We then have $t^2 - it - 1 = 0$. Therefore, by the quadratic formula:

$$t = \frac{i \pm \sqrt{(-i)^2 - 4(-1)(1)}}{2} = \frac{i \pm \sqrt{3}}{2}.$$

Therefore, $e^{i\theta} = \frac{\sqrt{3}}{2} + \frac{1}{2}i = e^{i(\pi/6+2k\pi)}, k \in \mathbb{Z}$ or $e^{i\theta} = \frac{-\sqrt{3}}{2} + \frac{1}{2}i = e^{i(5\pi/6+2k\pi)}, k \in \mathbb{Z}$, and so the possible values of $\sin^{-1}(1/2) = \theta$ are $\frac{\pi}{6} + 2k\pi, k \in \mathbb{Z}$ or $\frac{5\pi}{6} + 2k\pi, k \in \mathbb{Z}$.