Problem Set 4A: Assignment 4 – Solutions

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MAT 342 – Applied Complex Analysis
Summer Session II 2019

DUE: August 8th, 2019 – AT THE BEGINNING OF CLASS.

Exercise 0. Review everything you've studied this week before proceeding!

Exercise 1. Evaluate $\int_\gamma \frac{dz}{1+z^2}$ where γ is a circle of radius 2 and center 0.

Solution: The singularities of the integrand occur at $z = \pm i$; both of which are contained inside the path of integration. So we need to simplify the integral in such a way to isolate each one of them.

$$\begin{aligned}
\int_\gamma \frac{dz}{1+z^2} &= \int_\gamma \frac{1}{(z-i)(z+i)} dz \\
&= \int_\gamma \frac{1}{2i} \left(\frac{(z+i)-(z-i)}{(z+i)(z-i)} \right) dz \\
&= \frac{1}{2i} \int_\gamma \left(\frac{1}{z-i} - \frac{1}{z+i} \right) dz \\
&= \frac{1}{2i} \int_\gamma \frac{1}{z-i} dz - \frac{1}{2i} \int_\gamma \frac{1}{z+i} dz \\
&= \frac{1}{2i}(2\pi i) - \frac{1}{2i}(2\pi i),
\end{aligned}$$

by Cauchy's integral formula applied to each integral with the constant function $f(z) = 1$. (Or using the winding number.)

Exercise 2. Evaluate

$$\int_\gamma \frac{z^2 + e^z}{z(z-3)} dz$$

where γ is the unit circle.

Solution: The integrand has singularities at $z = 0$ and $z = 3$, but only $z = 0$ falls within the path of integration. Therefore, by defining $f(z) = \dfrac{z^2 + e^z}{z - 3}$ and using Cauchy's integral formula:

$$\int_\gamma \frac{z^2 + e^z}{z(z-3)} dz = \int_\gamma \frac{\frac{z^2 + e^z}{z-3}}{z} dz$$
$$= \int_\gamma \frac{f(z)}{z - 0} dz = 2\pi i f(0) = -\frac{2\pi i}{3}.$$

Exercise 3. Suppose that $|z| \leq 2$. Prove that the series

$$\sum_{n=1}^{\infty} \frac{2z^2}{n^2 + |z|}$$

converges absolutely.

Solution: We have:

$$\left| \frac{2z^2}{n^2 + |z|} \right| = \frac{|2z^2|}{|n^2 + |z||} = \frac{2|z|^2}{n^2 + |z|}.$$

Since $|z| \leq 2$, then $2|z|^2 \leq 8$. And since $|z| \geq 0$ in general, then: $n^2 + |z| \geq n^2$ and so $\dfrac{1}{n^2 + |z|} \leq \dfrac{1}{n^2}$. Therefore:

$$\left| \frac{2z^2}{n^2 + |z|} \right| \leq \frac{8}{n^2},$$

and so by letting $M_n = \dfrac{8}{n^2}$, it follows that the series converges absolutely by the Weierstraβ M-test.

Exercise 4. Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is convergent on some disk $\{|z| < R\}$ where $R > 0$. Suppose also that $f(z) = f(-z)$. Show that $a_n = 0$ for odd values of n; i.e. for $n = 1, 3, 5, 7, \cdots$.

Solution: We have

$$f(-z) = \sum_{n=1}^{\infty} a_n(-z)^n = \sum_{n=1}^{\infty} a_n(-1)^n z^n = \sum_{n=1}^{\infty} \left((-1)^n a_n\right) z^n,$$

meaning that the coefficients for the Taylor expansion of $f(-z)$ are $(-1)^n a_n$ and so if $f(z) = f(-z)$, then it must be the case that $a_n = (-1)^n a_n$ for all $n \geq 0$. For ood values of n, we have that $(-1)^n = -1$. Therefore, for odd values of n, $a_n = -a_n$; meaning that $a_n = 0$ for odd values of n.

Exercise 5. Let $f(z) = \dfrac{z^2 - 1}{\cos(\pi z) + 1}$ have the series expansion $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ near $z = 0$. Compute a_0, a_1 and a_2.

<u>Solution:</u> We know that $a_0 = f(0)$, $a_1 = \dfrac{f'(0)}{1!} = f'(0)$ and $a_2 = \dfrac{f''(0)}{2!} = \dfrac{f''(0)}{2}$.

Clearly, $a_0 = -\dfrac{1}{2}$. For a_1:

$$f'(z) = \frac{(z^2-1)'(\cos(\pi z)+1) - (\cos(\pi z)+1)'(z^2-1)}{(\cos(\pi z)+1)^2} = \frac{2z(\cos(\pi z)+1) - (-\pi\sin(\pi z))(z^2-1)}{(\cos(\pi z)+1)^2},$$

and so $a_1 = f'(0) = 0$.

For a_2:

$$\begin{aligned}
f''(z) &= \left(\frac{2z(\cos(\pi z)+1) - (-\pi\sin(\pi z))(z^2-1)}{(\cos(\pi z)+1)^2}\right)' \\
&= \left(\frac{2z\cos(\pi z) + 2z + \pi\sin(\pi z)(z^2-1)}{(\cos(\pi z)+1)^2}\right)' \\
&= \frac{(2z\cos(\pi z) + 2z + \pi\sin(\pi z)(z^2-1))'(\cos(\pi z)+1)^2}{(\cos(\pi z)+1)^4} \\
&\quad - \frac{\left((\cos(\pi z)+1)^2\right)'(2z\cos(\pi z) + 2z + \pi\sin(\pi z)(z^2-1))}{(\cos(\pi z)+1)^4} \\
&= \frac{(2\cos(\pi z) - 2\pi\sin(\pi z) + 2 + \pi^2\cos(\pi z)(z^2-1) + 2\pi z\sin(\pi z))(\cos(\pi z)+1)^2}{(\cos(\pi z)+1)^4} \\
&\quad - \frac{\left(2(-\pi\sin(\pi z))(\cos(\pi z)+1)\right)(2z\cos(\pi z) + 2z + \pi\sin(\pi z)(z^2-1))}{(\cos(\pi z)+1)^4} \\
&= \frac{(2+2+\pi^2(-1))(2^2)}{2^4} = \frac{4-\pi^2}{4},
\end{aligned}$$

and so $a_2 = \dfrac{\frac{4-\pi^2}{4}}{2} = \dfrac{4-\pi^2}{8} = \dfrac{1}{2} - \dfrac{\pi^2}{8}$.