Problem Set 3A: Assignment 3 – Solutions

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MAT 342 – Applied Complex Analysis
Summer Session II 2019

DUE: August 1st, 2019 – AT THE BEGINNING OF CLASS.

Exercise 0. Review everything you've studied this week before proceeding!

Exercise 1. Let $f(z) = \dfrac{e^{z^2}}{(2-z)^2}$.

1. What is the value of the integral of f over any simple closed curve inside the unit semi-circle?
2. What is the value of the integral of f over the circle of radius 2 centered at the origin?

Solution:

1. Clearly, f is holomorphic inside the unit disk (the region bounded by the unit circle) and on the unit circle. Therefore, the integral of f over any simply closed curve inside the unit semi-circle is 0.
2. Since 2 is inside the region bounded by the circle of radius 2 centered at 0, then by Cauchy's integral formula:

$$\int_{\{|z|=2\}} f(z)dz = \int_{\{|z|=2\}} \frac{e^{z^2}}{(z-2)^2}dz = \frac{2\pi i}{2!}\left(\frac{d}{dz}\bigg|_{z=2} \left(e^{z^2}\right)\right).$$

Differentiating e^{z^2}, we obtain:

$$\left(e^{z^2}\right)' = 2ze^{z^2},$$

and so the value of the integral is $\frac{2\pi i}{2!}(2(2)e^{2^2}) = 2e^4\pi i$.

Exercise 2. Suppose that f is holomorphic on the disk $\{|z| < 2\}$ and satisfies $\left|f(z)\right| < \sqrt{2}$. What estimate can be made about $\left|f'(0)\right|$?

Solution: By Cauchy's estimates: $\left|f'(0)\right| \leq \frac{1!}{2^1}\sqrt{2} = \frac{1}{\sqrt{2}}$.

Exercise 3. Evaluate the integral

$$\int_\gamma \left(\left(\mathrm{Re}(z)\right)^2 + i\left(\mathrm{Im}(z)\right)^2\right) dz,$$

where γ is the line joining 1 to i.

Solution: The line γ can be parametrized by $\gamma(t) = (1-t)(1)+ti = (1-t)+it, 0 \leq t \leq 1$. So then:

$$\begin{aligned}
\int_\gamma \left(\left(\mathrm{Re}(z)\right)^2 + i\left(\mathrm{Im}(z)\right)^2\right) dz &= \int_0^1 \left(\left(\mathrm{Re}(\gamma(t))\right)^2 + i\left(\mathrm{Im}(\gamma(t))\right)^2\right) \gamma'(t)dt \\
&= \int_0^1 \left((1-t)^2 + it^2\right)(-1+i)dt \\
&= (-1+i)\int_0^1 \left(1-2t+t^2+it^2\right) dt \\
&= (-1+i)\int_0^1 \left((1+i)t^2-2t+1\right) dt \\
&= (-1+i)\left[(1+i)\frac{t^3}{3} - t^2 + t\right]_{t=0}^{t=1} \\
&= (-1+i)\left(\frac{1+i}{3}\right) = \frac{i^2-1^2}{3} = -\frac{2}{3}.
\end{aligned}$$

Exercise 4. Let f be an entire function, and write $f(x+iy) = u(x,y) + iv(x,y)$. Suppose that $u(x,y) + v(x,y) \geq 1$. Let $\varphi(z) = e^{-(1-i)f(z)}$.

1. Is the function φ entire? Is it bounded?

2. What can you say about f based on your answer to 1?

Solution:

1. Since the exponential is entire and f is entire, then φ is entire. Let us estimate $\left|\varphi(z)\right|$:

$$\left|\varphi(z)\right| = \left|e^{-(1-i)f(x+iy)}\right| = \left|e^{-(1-i)(u(x,y)+iv(x,y)}\right| = \left|e^{-(u(x,y)+v(x,y))+i(-u(x,y)-v(x,y))}\right|,$$

and since

$$\left|e^{-(u(x,y)+v(x,y))+i(-u(x,y)-v(x,y))}\right| = e^{\mathrm{Re}\left(-(u(x,y)+v(x,y))+i(-u(x,y)-v(x,y))\right)} = e^{-(u(x,y)+v(x,y))}$$

and $u(x,y) + v(x,y) \geq 1$, it then follows that $\left|\varphi(z)\right| \leq e$. Therefore, φ is bounded.

2. By our answer to question 1, φ is entire and bounded. So φ is constant by Liouville's theorem. Hence $\log(\varphi) = -(1-i)f(z) + i2\pi n, n \in \mathbb{Z}$ is constant. Therefore, n must be a fixed integer n_0. And so f is constant.

Exercise 5. Prove that $g(z) = \sum_{n=1}^{\infty} \frac{1}{z^n}$ defines a holomorphic function for $|z| > 1$.

Solution: Let $g_n(z) = \frac{1}{z^n}$. Then, for $|z| > 1$, we have $\left|1/z\right| < 1$. This means that the series $\sum_{n=1}^{\infty} \left|\frac{1}{z}\right|^n$ is a geometric series. Therefore, it converges absolutely. Since the function $g_n(z) = \frac{1}{z^n}$ is holomorphic for $|z| > 1$ for each $n \geq 0$, then it follows that g is holomorphic by the Analytic Convergence Theorem.