Problem 1.

(a) Find the square roots of $2+i$. (1 point)

(b) Find all the values of $i^{1/4}$. (1 point)

Solution:

(a) Let $2+i=(a+ib)^2$ where $a,b\in\mathbb{R}$. Then: $a^2-b^2=2$ and $2ab=1$. So a and b must have the same sign since $ab>0$ and also $a^2=2+b^2$. Squaring the second equation, we obtain that $4a^2b^2=1$ and so substituting $2+b^2$ for a^2, we see that $4(2+b^2)b^2=1$, therefore: $4b^4+8b^2-1=0$. By the quadratic formula:

$$b^2=\frac{-8\pm\sqrt{8^2-4(4)(-1)}}{2(4)}=\frac{-8\pm\sqrt{64+16}}{8}=\frac{-8\pm4\sqrt{5}}{8}=-1\pm\frac{\sqrt{5}}{2},$$

but since b^2 must be positive, then it follows that $b^2=-1+\dfrac{\sqrt{5}}{2}$ and so $b=\pm\sqrt{-1+\dfrac{\sqrt{5}}{2}}$. But then: $a^2=2+\left(-1+\dfrac{\sqrt{5}}{2}\right)=1+\dfrac{\sqrt{5}}{2}$, and so $a=\pm\sqrt{1+\dfrac{\sqrt{5}}{2}}$, which implies that the square roots of $2+i$ are $\pm\sqrt{1+\dfrac{\sqrt{5}}{2}}\pm i\sqrt{-1+\dfrac{\sqrt{5}}{2}}$.

(b) By definition, $i^{1/4}=e^{(1/4)\log(i)}$. Clearly, $|i|=1$ and $\arg(i)=\pi/2$. Therefore, we see that $\log(i)=\ln(1)+i\pi/2+i2\pi n, n\in\mathbb{Z}=i\pi/2+i2\pi n, n\in\mathbb{Z}$. And so we obtain $i^{1/4}=e^{(1/4)(i\pi/2+i2\pi n)}=e^{i\pi/8+i\pi n/2}$ where $n\in\mathbb{Z}$.

Problem 2.

Let $z = x + iy$. Prove the following inequalities.

(a) $\left|e^z + e^{z^2}\right| \le e^x + e^{x^2-y^2}$. (1 point)

(b) $\left|e^{iz} + e^{iz^2}\right| \le e^{-y} + e^{-2xy}$. (1 point)

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Solution:

Let us express each of z^2, iz and iz^2 in terms of x and y.

- $z^2 = (x+iy)^2 = (x^2 - y^2) + i(2xy)$,
- $iz = i(x+iy) = ix - y = -y + ix$,
- $iz^2 = i\left((x^2-y^2) + i(2xy)\right) = i(x^2-y^2) - (2xy) = (-2xy) + i(x^2-y^2)$.

(a) Using the triangle inequality and the fact that $|e^w| = e^{\mathrm{Re}(w)}$ for any $w \in \mathbb{C}$:

$$\left|e^z + e^{z^2}\right| \le |e^z| + \left|e^{z^2}\right| = e^{\mathrm{Re}(z)} + e^{\mathrm{Re}(z^2)} = e^x + e^{x^2-y^2}.$$

(b) Again, using the triangle inequality and the fact that $|e^w| = e^{\mathrm{Re}(w)}$ for any $w \in \mathbb{C}$:

$$\left|e^{iz}z + e^{iz^2}\right| \le \left|e^{iz}\right| + \left|e^{iz^2}\right| = e^{\mathrm{Re}(iz)} + e^{\mathrm{Re}(iz^2)} = e^{-y} + e^{-2xy}.$$

Problem 3.

(a) Use the maximum modulus theorem to determine the maximum value of $\left|\frac{z}{z^2+9}\right|$ on the disk $\{|z| \leq 2\}$. (1 point)

(b) Using your answer to (a), conclude that

$$\left|\int_{\{|z|=2\}} \frac{z}{z^2+9} dz\right| \leq \frac{8\pi}{5}.$$

(1 point)
(The original bound which was $8\pi/\sqrt{65}$ was a mistake.)

(c) What is the actual value of $\left|\int_{\{|z|=2\}} \frac{z}{z^2+9} dz\right|$? (1 point)

Solution:

(a) The function $z \mapsto \frac{z}{z^2+9}$ is holomorphic on the disk $\{|z| \leq 2\}$ and clearly, it is not constant. Therefore, its modulus achieves its maximum value on the boundary: $\{|z| = 2\}$. When $|z| = 2$, we first have:

$$\left|\frac{z}{z^2+9}\right| = \frac{|z|}{|z^2+9|} = \frac{2}{|z^2+9|}.$$

Additionally, since $|z| = 2$, we may write $z = 2e^{it}, 0 \leq t \leq 2\pi$.
Therefore $z^2 + 9 = (2e^{it})^2 + 9 = 4e^{i2t} + 9 = (9 + 4\cos(2t)) + i4\sin(2t)$ and so:

$$|z^2+9| = \sqrt{(9+4\cos(2t))^2 + (4\sin(2t))^2} = \sqrt{81 + 72\cos(2t) + 16\cos^2(2t) + 16\sin^2(2t)}.$$

Therefore, on the boundary:

$$\left|\frac{z}{z^2+9}\right| = \frac{2}{\sqrt{82+18\cos(2t)}} = \frac{2}{= \sqrt{81+72\cos(2t)+16}} = \frac{2}{\sqrt{97+72\cos(2t)}}.$$

Clearly, this is maximized when $\cos(2t) = -1$. So the maximum value is $\frac{2}{\sqrt{25}} = \frac{2}{5}$.

(b) Since the length of the contour is 4π, then by the previous questions, the absolute value of the integral is less than or equal to $\frac{2}{5} \cdot 4\pi = \frac{8\pi}{5}$.

(c) The only singularities of $z^2 + 9$ occur at $z = \pm 3i$, and so $z \mapsto \frac{z}{z^2+9}$ is holomorphic inside the disk $\{|z| \leq 2\}$ and so the integral is equal to 0 by Cauchy's theorem. So its absolute value is also 0, obviously.

Problem 4.

Find conditions on a, b and c so that the function defined by
$f(z) = f(x+iy) = e^x \cos(ay) + ie^x \sin(y+b) + c$ is entire. (3 points)
Hint: Remember that $\cos(\alpha) = \cos(\beta)$ if and only if $\alpha = \pm\beta + 2\pi n; n \in \mathbb{Z}$.

Solution:

Let $u(x,y) = e^x \cos(ay)$ and $v(x,y) = e^x \sin(y+b)$.
The Cauchy-Riemann equations state that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$.
Therefore:

$$\frac{\partial}{\partial x}(e^x \cos(ay)) = \frac{\partial}{\partial y}(e^x \sin(y+b))$$

and

$$\frac{\partial}{\partial x}(e^x \sin(y+b)) = -\frac{\partial}{\partial y}(e^x \cos(ay)).$$

The first equation tells us that $e^x \cos(ay) = e^x \cos(y+b)$, and so $\cos(ay) = \cos(y+b)$.
Therefore: $ay = \pm(y+b) + 2\pi n = \pm y \pm b + 2\pi n, n \in \mathbb{Z}$. So far, this tells us that $a = \pm 1$ and $\pm b + 2\pi n = 0, n \in \mathbb{Z}$. Therefore, $a = \pm 1$ and $b = 2\pi n, n \in \mathbb{Z}$.
(The $\pm$ is unnecessary since $n \in \mathbb{Z}$.)
From the second Cauchy-Riemann equation, we obtain
$e^x \sin(y+b) = -e^x(-\sin(ay)) = e^x \sin(ay)$, so that $\sin(y+b) = \sin(ay)$.
Since $b = 2\pi n, n \in \mathbb{Z}$, $\sin(y+b) = \sin(y)$. Now $a = \pm 1$, but if $a = -1$, then
$\sin(y) = \sin(y+b) = \sin(ay) = \sin(-y) = -\sin(y)$, which is not true in general.
So a must be 1.
In conclusion: $a = 1$, $b = 2\pi n, n \in \mathbb{Z}$ and c can be any complex number.

Problem 5.

(a) Evaluate the residue of $\frac{z^2-1}{(z^2+1)^2}$ at $z_0 = i$. (1 point)

(b) Conclude the value of the integral

$$\int_\gamma \frac{z^2-1}{(z^2+1)^2} dz$$

where γ is the circle of radius 1 centered at i. (1 point)

(c) Use Cauchy's integral formula to compute in the integral in (b) and verify that your answer was correct. (1 point)

Solution:

(a) We may rewrite $\frac{z^2-1}{(z^2+1)^2}$ as:

$$\frac{z^2-1}{(z^2+1)^2} = \frac{z^2-1}{\big((z-i)(z+i)\big)^2} = \frac{z^2-1}{(z-i)^2(z+i)^2} = \frac{\frac{z^2-1}{(z+i)^2}}{(z-i)^2}.$$

Clearly, $z_0 = i$ is a double pole for this function, and so letting $\varphi(z) = \frac{z^2-1}{(z+i)^2}$, we have:

$$\operatorname{Res}\left(\frac{z^2-1}{(z^2+1)^2}, i\right) = \frac{\varphi'(i)}{(2-1)!},$$

and since

$$\varphi'(z) = \left(\frac{z^2-1}{(z+i)^2}\right)' = \frac{2z(z+i)^2 - 2(z+i)(z^2-1)}{(z+i)^4} = \frac{2z(z+i) - 2(z^2-1)}{(z+i)^3},$$

then:

$$\operatorname{Res}\left(\frac{z^2-1}{(z^2+1)^2}, i\right) = \frac{2i(i+i) - 2(i^2-1)}{(i+i)^3} = \frac{(2i)^2 - 2(-1-1)}{(2i)^3} = 0.$$

(b) By the Residue Theorem and our response to (a), the value of the integral is 0.

(c) Let us rewrite the integral:

$$\int_\gamma \frac{z^2-1}{(z^2+1)^2} dz = \int_\gamma \frac{\varphi(z)}{(z-i)^2} dz,$$

where $\varphi(z) = \frac{z^2-1}{(z+i)^2}$. By Cauchy's integral formula, since i is enclosed by γ, then the value of the integral is $\frac{2\pi i}{1!}\varphi'(i) = 0$.

Problem 6.

Consider the function defined by $f(z) = \dfrac{2\sin(z)}{e^z - 1}$.

(a) Which type of singularity does f have at $z_0 = 0$? Justify your answer using **two** different methods. (2 points)

(b) What is the residue of f at $z_0 = 0$? (1 point)

Solution:

(a) Recall that around $z_0 = 0$:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + \cdots$$

and

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$

Therefore:

$$f(z) = \frac{2\left(z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} + \cdots\right)}{\left(1 + z + \dfrac{z^2}{2} + \cdots\right) - 1} = \frac{2\left(z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} + \cdots\right)}{z + \dfrac{z^2}{2} + \cdots} = \frac{2z\left(1 - \dfrac{z^2}{3!} + \dfrac{z^4}{5!} + \cdots\right)}{z\left(1 + \dfrac{z}{2!} + \cdots\right)}.$$

Clearly, the numerator has a zero of order 1 at $z_0 = 0$ and the denominator has a zero of order 1 at $z_0 = 0$. So f has a removable singularity at $z_0 = 0$.

The second way to see this is by showing that the limit $\lim_{z\to 0}(z - 0)f(z)$ equals 0:

$$\lim_{z\to 0}(z-0)f(z) = \lim_{z\to 0} z\frac{2\sin(z)}{e^z - 1} = \lim_{z\to 0} \frac{2\sin(z)}{\dfrac{e^z - 11}{z}} = 2\left(\frac{\lim_{z\to 0}\sin(z)}{\lim_{z\to 0}\dfrac{e^z - e^0}{z - 0}}\right) = 2\left(\frac{\sin(0)}{\exp'(0)}\right) = 2\left(\frac{0}{1}\right) = 0.$$

(b) Since the singularity is removable, the residue is 0.

Problem 7.

(a) What is the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{w^n}{(3+i)^n}$? (1 point)

(b) Using an appropriate substitution, find for which values of $|z|$, the series $\sum_{n=0}^{\infty} \frac{z^{3n}}{(3+i)^n}$ converges, and for which values of $|z|$ it diverges. (1 point)

(c) Show that the series $\sum_{n=0}^{\infty} \frac{z^{3n}}{(3+i)^n}$ defines a holomorphic function wherever it converges. What is this holomorphic function? (2 points)
Hint: Think of using the geometric series.

Solution:

(a) The coefficients of the power series are $a_n = \frac{1}{(3+i)^n}$ for $n \geq 0$. We may compute the radius of convergence using the ratio test definition:

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{(3+i)^n}}{\frac{1}{(3+i)^{n+1}}}\right| = \lim_{n\to\infty}\left|\frac{(3+i)^{n+1}}{(3+i)^n}\right| = \lim_{n\to\infty}|3+i| = |3+i| = \sqrt{10}.$$

(b) The series in (a) converges for $|w| < 10^{1/2}$ and diverges for $|w| > 10^{1/2}$. Substituting $w = z^3$ in the series in (a), we obtain the series in (b) and so the series in (b) converges for $\left|z^3\right| < 10^{1/2}$ and diverges for $\left|z^3\right| > 10^{1/2}$. Therefore, the series in (b) converges for $|z| < 10^{1/6}$ and diverges for $|z| > 10^{1/6}$.

(c) Clearly, since the functions composing the series are holomorphic, the series converges (when it does) to a holomorphic function by the analytic convergence theorem.
Now note that we may rewrite the series as follows.

$$\sum_{n=0}^{\infty} \frac{z^{3n}}{(3+i)^n} = \sum_{n=0}^{\infty}\left(\frac{z^3}{3+i}\right)^n.$$

By our answer to (b), this series converges exactly when $\left|z^3\right| < 10^{1/2}$. Therefore $\left|\frac{z^3}{3+i}\right| = \frac{\left|z^3\right|}{10^{1/2}} < 1$, and so our series defines a geometric series. Rewriting $t = \frac{z^3}{3+i}$, we have:

$$\sum_{n=0}^{\infty} \frac{z^{3n}}{(3+i)^n} = \sum_{n=0}^{\infty} t^n = \frac{1}{1-t} = \frac{1}{1-\frac{z^3}{3+i}}.$$

Problem 8. – BONUS

Evaluate the following integral using **two** different methods.

$$\int_\gamma \frac{e^{z^2}\sin(z)}{z-z^3}dz,$$

where γ is the circle of radius 2 centered at 0. (6 points)

Hint: Think of rewriting $\frac{1}{z-z^3}$ using partial fractions.

Solution:

We have: $\frac{e^{z^2}\sin(z)}{z-z^3} = \frac{e^{z^2}\sin(z)}{z(1-z^2)} = \frac{e^{z^2}\sin(z)}{z(1-z)(1+z)}$, and so the singularities of the function $z \mapsto \frac{e^{z^2}\sin(z)}{z-z^3}$ occur at 0 and ± 1; all of which are enclosed by the path of integration.

Using partial fractions:

$$\frac{1}{z(1-z)(1+z)} = \frac{A}{z} + \frac{B}{1-z} + \frac{C}{1+z} = \frac{A(1-z)(1+z) + Bz(1+z) + Cz(1-z)}{z(1-z)(1+z)},$$

and so: $1 = A(1-z)(1+z) + Bz(1+z) + Cz(1-z)$. Plugging in $z = 0$, we obtain $1 = A$. Plugging in $z = 1$, we obtain $1 = 2B$ and so $B = 1/2$. Plugging in $z = -1$, we obtain $1 = C(-1)(2)$ and so $C = -1/2$. Therefore:

$$\frac{1}{z(1-z)(1+z)} = \frac{1}{z} + \frac{1/2}{1-z} + \frac{-1/2}{1+z} = \frac{1}{z} - \frac{1}{2}\frac{1}{z-1} - \frac{1}{2}\frac{1}{z+1}.$$

Hence, rewriting and using Cauchy's integral formula:

$$\begin{aligned}\int_\gamma \frac{e^{z^2}\sin(z)}{z-z^3} &= \int_\gamma \frac{e^{z^2}\sin(z)}{z}dz - \frac{1}{2}\int_\gamma \frac{e^{z^2}\sin(z)}{z-1}dz - \frac{1}{2}\int_\gamma \frac{e^{z^2}\sin(z)}{z+1}dz \\ &= 2\pi i e^{0^2}\sin(0) - \frac{1}{2}(2\pi i e^{1^2}\sin(1)) - \frac{1}{2}(2\pi i e^{(-1)^2}\sin(-1)) = 0.\end{aligned}$$

Otherwise, we may evaluate this integral using the Residue Theorem. At $z_0 = 0$, e^{z^2} does not vanish but $\sin(z)$ does. We know that $\sin(z)$ has a zero of order 1 at $z_0 = 0$ from its power expansion. Since the denominator of the function has a zero of order 1 at $z_0 = 0$, then it follow that this function has a removable singularity at $z_0 = 0$.

Therefore, its residue at $z_0 = 0$ is 0.

Clearly, this function has simple poles at $z = \pm 1$ and so the residues at those points are:

$$\text{Res}\left(\frac{e^{z^2}\sin(z)}{z-z^3}; 1\right) = \lim_{z\to 1}(z-1)\left(\frac{e^{z^2}\sin(z)}{z(1-z)(1+z)}\right) = \lim_{z\to 1} -\frac{e^{z^2}\sin(z)}{z(1+z)} = -\frac{e}{2}\sin(1).$$

$$\text{Res}\left(\frac{e^{z^2}\sin(z)}{z-z^3}; -1\right) = \lim_{z\to -1}(z+1)\left(\frac{e^{z^2}\sin(z)}{z(1-z)(1+z)}\right) = \lim_{z\to -1} \frac{e^{z^2}\sin(z)}{z(1-z)} = -\frac{e}{2}\sin(-1).$$

Summing up the residues after multiplying each one of them by $2\pi i$, we obtain the same result: 0.