Problem 1.

1. Find all the values of:
   (a) $\left(1+i\sqrt{3}\right)^{1-i}$.
   (b) $\log\left((1+i)^{2i}\right)$.
2. Find all the possible values of $\sin^{-1}(i)$.

Solution:

1. (a) By definition:

$$\left(1+i\sqrt{3}\right)^{1-i} = e^{(1-i)\log\left(1+i\sqrt{3}\right)}.$$

Let us first compute $\log\left(1+i\sqrt{3}\right)$. In order to do so, we need to find the modulus of $1+i\sqrt{3}$ and its argument. We have $\left|1+i\sqrt{3}\right| = 2$, and so:

$$1+i\sqrt{3} = 2\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) = 2\left(\cos(\pi/3)+i\sin(\pi/3)\right).$$

Therefore:

$$\log\left(1+i\sqrt{3}\right) = \ln(2)+i\pi/3+i2\pi n, n\in\mathbb{Z}.$$

And so:

$$\begin{aligned}\left(1+i\sqrt{3}\right)^{1-i} &= e^{(1-i)(\ln(2)+i\pi/3+i2\pi n)}, n\in\mathbb{Z}\\ &= e^{\ln(2)+i\pi/3+i2\pi n-i\ln(2)+\pi/3+2\pi n}, n\in\mathbb{Z}\\ &= e^{(\ln(2)+\pi/3+2\pi n)}e^{i\left(-\ln(2)+\pi/3\right)}e^{i2\pi n}, n\in\mathbb{Z}\\ &= 2e^{\pi/3}e^{2\pi n}e^{i\left(\pi/3-\ln(2)\right)}, n\in\mathbb{Z}.\end{aligned}$$

0.25 for the modulus, 0.25 for the argument, 0.25 for the expression of the log, and 0.25 for the final answer. Total for 1. (a): 1 point.

(b) We first need to simplify $(1+i)^{2i}$:

$$(1+i)^{2i} = e^{2i\log(1+i)}.$$

We have that $|1+i| = \sqrt{2}$ and so:

$$1+i = \sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right) = \sqrt{2}(\cos(\pi/4)+i\sin(\pi/4)).$$

Therefore:

$$\log(1+i) = \ln\left(\sqrt{2}\right)+i\pi/4+i2\pi n = \frac{1}{2}\ln(2)+i\pi/4+i2\pi n, n\in\mathbb{Z}.$$

$$
\begin{aligned}
(1+i)^{2i} &= e^{2i\log(1+i)} \\
&= e^{2i\left(\frac{1}{2}\ln(2)+i\pi/4+i2\pi n\right)}, n\in\mathbb{Z} \\
&= e^{i\ln(2)-\pi/2-4\pi n}, n\in\mathbb{Z} \\
&= \left(e^{-(\pi/2+4\pi n)}\right)\left(e^{i\ln(2)}\right), n\in\mathbb{Z}.
\end{aligned}
$$

Therefore: $\left|(1+i)^{2i}\right| = e^{-(\pi/2+4\pi)}, n\in\mathbb{Z}$ and $\arg\left((1+i)^{2i}\right)=\ln(2)$; and so:

$$\log\left((1+i)^{2i}\right) = \ln\left(e^{-(\pi/2+4\pi n)}\right) + i\ln(2) + i2\pi k = -\pi/2-4\pi n + i\ln(2) + i2\pi k,$$

where $k,n\in\mathbb{Z}$.

0.25 for the modulus for the first log, 0.25 for the argument for the first log, 0.25 for the correct expression of the first log, 0.25 for the modulus of the second log, 0.25 for the argument of the second log, and 0.25 for the final result. Total for 1. (b): 1.5 points.

2. Let $\theta=\sin^{-1}(i)$.

$$
\begin{aligned}
\theta=\sin^{-1}(i) &\iff \sin(\theta)=i \\
&\iff \frac{e^{i\theta}-e^{-i\theta}}{2i}=i \\
&\iff e^{i\theta}-e^{-i\theta}=(2i)(i)=-2 \\
&\iff \left(e^{i\theta}\right)^2-1=-2e^{i\theta} \\
&\iff \left(e^{i\theta}\right)^2+2e^{i\theta}-1=0.
\end{aligned}
$$

By the quadratic formula:

$$e^{i\theta}=\frac{-2\pm\sqrt{2^2-4(1)(-1)}}{2(1)}=\frac{-2\pm\sqrt{8}}{2}=\frac{-2\pm2\sqrt{2}}{2}=-1\pm\sqrt{2}.$$

Therefore: $i\theta=\log\left(-1\pm\sqrt{2}\right)$ so that $\theta=-i\log\left(-1\pm\sqrt{2}\right)$.
Now $\left|-1\pm\sqrt{2}\right|=\sqrt{2}\pm1$ and since $-1\pm\sqrt{2}\in\mathbb{R}$, it follows that its argument must be a multiple of π; i.e. $\arg\left(-1\pm\sqrt{2}\right)=k\pi, k\ in\mathbb{Z}$. Therefore:

$$\sin^{-1}(i)=-i\left(\ln\left(\sqrt{2}\pm1\right)+ik\pi+i2\pi n\right)=-i\ln\left(\sqrt{2}\pm1\right)+k\pi, k\in\mathbb{Z}.$$

That last multiple of $2\pi i$ was dropped simply because adding a multiple of πi to a multiple of $2\pi i$ amounts to simply adding a multiple of πi.

0.25 for the correct formula for the complex sine, 0.25 for setting up the quadratic equation correctly, 0.25 for solving it correctly, 0.25 for the correct modulus of the log, 0.25 for its argument and 0.25 for the correct final answer. Total for 2: 1.5 points.

Problem 2.

Define the complex tangent by $\tan(w) = \dfrac{\sin(w)}{\cos(w)}$, whenever $\cos(w) \neq 0$.

Prove the identity:

$$z = \tan\left(\frac{1}{i}\log\left(\frac{1+iz}{1-iz}\right)^{1/2}\right).$$

Hint: Write $z = \tan(w)$ and solve for w.

Solution:

Let us start by writing down an expression for $\tan(w)$ in terms of complex exponentials.

$$\tan(w) = \frac{\sin(w)}{\cos(w)} = \frac{\dfrac{e^{iw}-e^{-iw}}{2i}}{\dfrac{e^{iw}+e^{-iw}}{2}} = \frac{1}{i}\frac{e^{iw}-e^{-iw}}{e^{iw}+e^{-iw}} = \frac{1}{i}\frac{e^{iw}-\dfrac{1}{e^{iw}}}{e^{iw}+\dfrac{1}{e^{iw}}} = \frac{1}{i}\frac{\dfrac{\left(e^{iw}\right)^2-1}{e^{iw}}}{\dfrac{\left(e^{iw}\right)^2+1}{e^{iw}}},$$

and so: $\tan(w) = \dfrac{1}{i}\dfrac{\left(e^{iw}\right)^2-1}{\left(e^{iw}\right)^2+1}$.

Now let us set $z = \tan(w)$ and solve for w.

$$\begin{aligned}
z = \tan(w) &\iff z = \frac{1}{i}\frac{\left(e^{iw}\right)^2-1}{\left(e^{iw}\right)^2+1}\\
&\iff (iz)\left(\left(e^{iw}\right)^2+1\right) = \left(e^{iw}\right)^2-1\\
&\iff (iz)\left(e^{iw}\right)^2 + iz = \left(e^{iw}\right)^2-1\\
&\iff (iz-1)\left(e^{iw}\right)^2 = -iz-1\\
&\iff \left(e^{iw}\right)^2 = \frac{-iz-1}{iz-1} = \frac{iz+1}{-iz+1} = \frac{1+iz}{1-iz}\\
&\iff e^{iw} = \left(\frac{1+iz}{1-iz}\right)^{1/2}\\
&\iff iw = \log\left(\left(\frac{1+iz}{1-iz}\right)^{1/2}\right)\\
&\iff w = \frac{1}{i}\log\left(\left(\frac{1+iz}{1-iz}\right)^{1/2}\right).
\end{aligned}$$

Therefore:

$$z = \tan\left(\frac{1}{i}\log\left(\left(\frac{1+iz}{1-iz}\right)^{1/2}\right)\right).$$

1 for the correct expression of the complex tangent function, 1 for isolating the square of e^{iw} and 1 for solving for w correctly. Total: 3 points.

Problem 3.

Show that:

$$\left|\int_{\{|z-1|=1\}} \frac{e^z}{z+1} dz\right| \leq 2\pi e^2.$$

Hint: You may want to parametrize the circle $\{|z-1|=1\}$ first.

Solution:

The circle $\{|z-1|=1\}$ may be parametrized as $z(t) = 1 + e^{it}, 0 \leq t \leq 2\pi$. So then:
$z(t) = 1 + \cos(t) + i\sin(t) = (1+\cos(t)) + i\sin(t)$.

We need to bound $\left|\frac{e^z}{z+1}\right|$ from above.

Therefore, we need to bound e^z from above and $z+1$ from below.

For e^z: $|e^z| = e^{\text{Re}(z)} = e^{1+\cos(t)} \leq e^{1+1} = e^2$.

Note that $z+1 = 1 + (1+\cos(t)) + i\sin(t) = (2+\cos(t)) + i\sin(t)$. Therefore:

$$\begin{aligned}
|z+1| = \sqrt{(2+\cos(t))^2 + (\sin(t))^2} &= \sqrt{4 + 4\cos(t) + \cos^2(t) + \sin^2(t)} \\
&= \sqrt{4 - 4\cos(t) + 1} \\
&= \sqrt{5 - 4\cos(t)} \\
&\geq \sqrt{5-4} = 1
\end{aligned}$$

Therefore, on the circle:

$$\left|\frac{e^z}{z+1}\right| \leq e^2,$$

and since the length of this circle is clearly 2π, we obtain the desired inequality.

0.5 for the parametrization, 1 for bounding the numerator, 1 for bounding the denominator, and 0.5 for proving the inequality. Total: 3 points.

Problem 4.

Consider the function $z \mapsto ze^z$.

1. Is it entire (holomorphic in all of $\mathbb{C}$)? Justify your answer either by invoking the basic properties of holomorphicity, or by making use of the limit-definition of holomorphicity.

2. Verify your answer to the first question using the Cauchy-Riemann equations.

Solution:

1. Since the functions $z \mapsto z$ and $z \mapsto e^z$ are entire functions, their product must be an entire function by the basic properties of holomorphic functions.
   If we use the definition instead, then choosing an arbitrary $z_0 \in \mathbb{C}$, we have:

$$\begin{aligned}
\lim_{z \to z_0} \frac{ze^z - z_0 e^{z_0}}{z - z_0} &= \lim_{z \to z_0} \frac{ze^z - z_0 e^z + z_0 e^z - z_0 e^{z_0}}{z - z_0} \\
&= \lim_{z \to z_0} \frac{(z - z_0)e^z + z_0(e^z - e^{z_0})}{z - z_0} \\
&= \lim_{z \to z_0} \frac{(z - z_0)e^z}{z - z_0} + \lim_{z \to z_0} \frac{z_0(e^z - e^{z_0})}{z - z_0} \\
&= \lim_{z \to z_0} e^z + z_0 \left(\lim_{z \to z_0} \frac{e^z - e^{z_0}}{z - z_0} \right) \\
&= e^{z_0} + z_0 \left(\lim_{z \to z_0} \frac{e^z - e^{z_0}}{z - z_0} \right).
\end{aligned}$$

   Since the function $z \mapsto e^z$ is entire, the limit multiplying z_0 always exists, and so the limit with which started always exists for any z_0. Therefore, the function $z \mapsto ze^z$ is holomorphic at any $z_0 \in \mathbb{C}$ meaning that it must be entire.

   1 point for pointing for $z \mapsto z$ is entire, 1 point for $z \mapsto e^z$ is entire, and 1 point for invoking that the product of entire functions is entire. If you did the limit-definition instead: 1 point for the correct limit, 1 point for simplifying the limit, and 1 point for justifying it exists. Either way, total for 1: 3 points.

2. Let us write $z = x + iy$, then:

$$\begin{aligned}
ze^z = (x + iy)\left(e^x(\cos(y) + i\sin(y))\right) &= (x + iy)(e^x \cos(y) + ie^x \sin(y)) \\
&= xe^x \cos(y) - ye^x \sin(y) + i(ye^x \cos(y) + xe^x \sin(y)),
\end{aligned}$$

   and so $u(x, y) = xe^x \cos(y) - ye^x \sin(y)$ and $v(x, y) = ye^x \cos(y) + xe^x \sin(y)$ are, respectively, the real and imaginary parts of the function $z \mapsto ze^z$. Let us compute the partial derivatives:

   - $\frac{\partial u}{\partial x} = (x + 1)e^x \cos(y) - y\sin(y)e^x.$
   - $\frac{\partial v}{\partial y} = e^x(\cos(y) - y\sin(y)) + xe^x \cos(y) = (x + 1)e^x \cos(y) - y\sin(y)e^x = \frac{\partial u}{\partial x}.$
   - $\frac{\partial u}{\partial y} = -xe^x \sin(y) - (e^x \sin(y) + ye^x \cos(y)) = -(x + 1)e^x \sin(y) - ye^x \cos(y).$
   - $\frac{\partial v}{\partial x} = y\cos(y)e^x + (x + 1)e^x \sin(y) = -\frac{\partial u}{\partial y}.$

The second and fourth lines show that the Cauchy-Riemann equations.

0.5 for finding the real part of the function, 0.5 for finding its imaginary part, 0.5 for the computation of each partial derivative and 0.5 for each Cauchy-Riemann equation. Total for 2: 4 points.

Problem 5.

Evaluate the following integrals using Cauchy's integral formulas.

1. $\displaystyle\int_{\{|z|=2\}} \frac{z^n}{z-1} dz, n \geq 0.$
2. $\displaystyle\int_{\{|z|=2\}} \frac{\sin(z)}{z} dz.$
3. $\displaystyle\int_{\{|z|=1\}} \frac{e^z}{z^4} dz.$

Solution:

1. Since 1 is inside the circle $\{|z| = 2\}$ and since $z \mapsto z^n$ is an entire function for all $n \geq 0$, it follows that, for any $n \geq 0$,
$$\int_{\{|z|=2\}} \frac{z^n}{z-1} dz = 2\pi i (1)^n = 2\pi i,$$
by Cauchy's integral formula.

   0.5 for using Cauchy's integral formula, 0.5 for the result.

2. Since 0 is inside the circle $\{|z| = 2\}$ and since $\sin(z)$ is an entire function, it follows that
$$\int_{\{|z|=2\}} \frac{\sin(z)}{z} dz = \int_{\{|z|=2\}} \frac{\sin(z)}{z-0} dz = 2\pi i \sin(0) = 0,$$
by Cauchy's integral formula.

   Same as above.

3. Let $f(z) = e^z$. Clearly, f is entire and since 0 is in the circle $\{|z| = 1\}$, then by Cauchy's general integral formula:
$$\int_{\{|z|=1\}} \frac{e^z}{z^4} dz = \int_{\{|z|=1\}} \frac{f(z)}{(z-0)^4} dz = \frac{2\pi i}{3!} f'''(0).$$
But $f'''(z) = e^z$ and so the value of the integral $\dfrac{2\pi i}{3!} = \dfrac{\pi i}{3}$.

   0.5 for using Cauchy's general integral formula, 0.5 for the correct result.

Problem 6. – BONUS

Let γ define the circle centered at 0 with radius 3. Knowing that 2 is one of the roots of $z^3 - 10z^2 + 32z - 32$, evaluate the integral

$$\int_\gamma \frac{2z^2 - 15z + 30}{z^3 - 10z^2 + 32z - 32} dz.$$

Hint: Think of using partial fractions.

Solution:

Let us try to factor the denominator since we know 2 is one of its roots. Since it is a polynomial of third degree, we may factor it as:

$$z^3 - 10z^2 + 32z - 32 = (z-2)(z-a)(z-b) = (z-2)(z^2 - (a+b)z + ab),$$

where $a, b \in \mathbb{C}$.

Distributing further:

$$z^3 - 10z^2 + 32z - 32 = z^3 - 2z^2 - (a+b)z^2 + 2(a+b)z + abz - 2ab,$$

and so: $-(2+a+b) = -10, 32 = ab + 2(a+b)$ and $-2ab = -32$. From the third equation, we know that $ab = 16$ and so they must have the same sign. Clearly none of a or b can be zero since 0 is not a root. Replacing $ab = 16$ in the second equation, $32 = 16 + 2(a+b)$ from which we obtain that $a + b = 8$. (Replacing the latter in the first equation only verifies that our findings so far are correct.) We have that $a + b = 8$ and $ab = 16$, and so $a = b = 4$. (We may obtain this by replacing $b = 4 - a$ in $ab = 16$, solving for b and then solving for a.)

Therefore: $z^3 - 10z^2 + 32z - 32 = (z-2)(z-4)^2$.

(1 point for setting up the system of equations for a and b. 1 point for a. 1 point for b. Total: 3 points.)

The point 2 falls within the region bounded by the path of integration but 4 does not.

(1 point for mentioning this.)

Therefore, we may rewrite the integral as

$$\int_\gamma \frac{2z^2 - 15z + 30}{z^3 - 10z^2 + 32z - 32} dz = \int_\gamma \frac{2z^2 - 15z + 30}{(z-2)(z-4)^2} dz = \int_\gamma \frac{f(z)}{z-2} dz,$$

where: $f(z) = \dfrac{2z^2 - 15z + 30}{(z-4)^2}$.

(1 points for rewriting the integral this way.)

Therefore, by Cauchy's integral formula:

$$\int_\gamma \frac{2z^2 - 15z + 30}{z^3 - 10z^2 + 32z - 32} dz = \int_\gamma \frac{f(z)}{z-2} dz = 2\pi i f(2) = (2\pi i)\left(\frac{2(2)^2 - 15(2) + 30}{(2-4)^2}\right) = 4\pi i.$$

(1 point for the correct final result.)

Total: 6 possible bonus points.