Problem 1.

For any propositions P and Q, write the truth tables for each of the propositions '$P \Rightarrow \neg Q$' and '$\neg(P \wedge Q)$'. Are these two propositions ('$P \Rightarrow \neg Q$' and '$\neg(P \wedge Q)$') equivalent?

Solution:

P	Q	$\neg Q$	$P \Rightarrow \neg Q$	$P \wedge Q$	$\neg(P \wedge Q)$
T	T	F	F	T	F
F	F	T	T	F	T
T	F	T	T	F	T
F	T	F	T	F	T

Since '$P \Rightarrow \neg Q$' and '$\neg(P \wedge Q)$' have the same truth values, they must be equivalent.

Problem 2. We define an integer m to be the *largest* number if it is larger (bigger) than any other *integer*.

(a) Rewrite the sentence 'm is larger than any other integer' in mathematical symbols using quantifiers if necessary.

(b) Prove that there exists no such number via a proof by contradiction.

Solution:

(a) $\forall n \in \mathbb{Z} : m > n$

(b) If there exists such a number m, then $m > n$ for *any* $n \in \mathbb{Z}$.
So for $n = m + 1$, for instance, $m > m + 1$ – which is impossible. Contradiction. Therefore, there exists no such number.

Problem 3. Prove by induction that $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$ for all $n \geq 1$.

Solution:

- **Base case:** For $n = 1$, we have:

$$\sum_{i=1}^{\mathbf{1}} \frac{1}{i(i+1)} = \frac{1}{1(1+1)} = 1 - \frac{1}{2} = \frac{\mathbf{1}}{\mathbf{1}+1}$$

So the base case holds.

- **Inductive step:** Suppose now as an inductive hypothesis that

$$\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$$

for some $k \geq 1$. Then:

$$\sum_{i=2}^{k+1} \frac{1}{i(i+1)} = \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{1}{k+1}\left(k + \frac{1}{k+2}\right)$$

And so:

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{1}{k+1}\left(\frac{k(k+2)+1}{k+2}\right) = \frac{1}{k+1} \cdot \frac{k^2+2k+1}{k+2} = \frac{1}{k+1} \cdot \frac{(k+1)^2}{k+1} = \frac{k+1}{k+2} = \frac{k+1}{(k+1)+1}$$

as required.

- **Conclusion:** By induction, $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$ for all $n \geq 1$.

Problem 4. Define the function $f : \mathbb{Q}^2 \longrightarrow \mathbb{Q}^2$ by $f(a, b) = (a + b, a - b)$.

(a) Prove that f is injective.

(b) Is f bijective? If so, prove it. If not, explain why and provide a counter-example.

Solution:

(a) Let $(a_1, b_1), (a_2, b_2) \in \mathbb{Q}^2$. Then:

$$\begin{aligned} f(a_1, b_1) = f(a_2, b_2) &\Rightarrow (a_1 + b_1, a_1 - b_1) = (a_2 + b_2, a_2 - b_2) \\ &\Rightarrow a_1 + b_1 = a_2 + b_2 \text{ and } a_1 - b_1 = a_2 - b_2 \\ &\Rightarrow (a_1 + b_1) + (a_1 - b_1) = (a_2 + b_2) + (a_2 - b_2) \\ &\text{and } (a_1 + b_1) - (a_1 - b_1) = (a_2 + b_2) - (a_2 - b_2) \\ &\Rightarrow 2a_1 = 2a_2 \text{ and } 2b_1 = 2b_2 \\ &\Rightarrow a_1 = a_2 \text{ and } b_1 = b_2 \\ &\Rightarrow (a_1, b_1) = (a_2, b_2) \end{aligned}$$

Therefore, f is injective.

(b) Since f is injective, it suffices to check whether f is surjective in order to check whether it is bijective. For f to be injective, we need to check the following:

$$\forall (x_1, x_2) \in \mathbb{Q}^2, \exists (y_1, y_2) \in \mathbb{N}^2 : f(x_1, x_2) = (y_1, y_2)$$

Now:

$$\begin{aligned} f(x_1, x_2) = (y_1, y_2) &\Leftrightarrow (x_1 + x_2, x_1 - x_2) = (y_1, y_2) \\ &\Leftrightarrow x_1 + x_2 = y_1 \text{ and } x_1 - x_2 = y_2 \\ &\Leftrightarrow (x_1 + x_2) + (x_1 - x_2) = y_1 + y_2 \text{ and } (x_1 + x_2) - (x_1 - x_2) = y_1 - y_2 \\ &\Leftrightarrow 2x_1 = y_1 + y_2 \text{ and } 2x_2 = y_1 - y_2 \\ &\Leftrightarrow x_1 = \frac{y_1 + y_2}{2} \text{ and} x_2 = \frac{y_1 - y_2}{2} \end{aligned}$$

As $y_1 \in \mathbb{Q}$ and $y_2 \in \mathbb{Q}$, $x_1 = \dfrac{y_1 + y_2}{2}$ and $x_2 = \dfrac{y_1 - y_2}{2}$ are also in $\mathbb{Q}$. Therefore, the equation $f(x_1, x_2) = (y_1, y_2)$ has a solution $(x_1, x_2) \in \mathbb{Q}^2$ for all $(y_1, y_2) \in \mathbb{Q}^2$.
So f is surjective. Since f is injective by (a), f is bijective.

Problem 5.

Let A, B and C. Using a double inclusion argument, prove that:

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

Solution:

$$\begin{aligned}
(x, y) \in A \times (B \cap C) &\Leftrightarrow [x \in A] \wedge [y \in (B \cap C)] \\
&\Leftrightarrow [x \in A] \wedge [[y \in B] \wedge [y \in C]] \\
&\Leftrightarrow [[x \in A] \wedge [y \in B]] \wedge [[x \in A] \wedge [y \in C]] \\
&\Leftrightarrow [(x, y) \in (A \times B)] \wedge [(x, y) \in (A \times C)] \\
&\Leftrightarrow (x, y) \in (A \times B) \cap (A \times C)
\end{aligned}$$

Therefore, by the definition of set equality, the result follows.

Problem 6.

Define the characteristic function of a set A as being the function such that:

$\chi_A(x) = 0$ if $x \notin A$ and $\chi_A(x) = 1$ if $x \in A$.

(a) Prove that $\chi_A(x) = 1 - \chi_{A^c}(x)$ for any set A. A^c denotes the complement of the set A.

(b) Prove that $\chi_{A\cap B}(x) = \chi_A(x)\chi_B(x)$ for any sets A and B.

(c) Prove that $\chi_{A\cup B}(x) = \chi_A(x) + \chi_B(x) - \chi_A(x)\chi_B(x)$ for any sets A and B.

Solution:

(a) By definition, $\chi_{A^c}(x) = 0$ if $x \notin A^c$ and $\chi_{A^c}(x) = 1$ if $x \in A^c$.
So $\chi_{A^c}(x) = 0$ if $x \in A$ and $\chi_{A^c}(x) = 1$ if $x \notin A$.
Thus: $1 - \chi_{A^c}(x) = 1 = \chi_A(x)$ if $x \in A$ and $1 - \chi_{A^c}(x) = 0 = \chi_A(x)$ if $x \notin A$.
Conclusion: $\chi_A(x) = 1 - \chi_{A^c}(x)$.

(b) $\chi_{A\cap B}(x) = 1$ if $x \in A \cap B$ and $\chi_{A\cap B}(x) = 0$ if $x \notin A \cap B$.
Now if $x \in A$, $\chi_A(x)\chi_B(x) = \chi_B(x)$ and $\chi_B(x) = 1$ if $x \in B$. So if $x \in A$ and $x \in B$, $\chi_A(x)\chi_B(x) = 1$ and $\chi_A(x)\chi_B(x) = 0$ otherwise. In other words, if $x \in A \cap B$, $\chi_A(x)\chi_B(x) = 1$ and $\chi_A(x)\chi_B(x) = 0$ if not. Therefore, $\chi_A(x)\chi_B(x) = 1$ if $x \in A \cap B$ and $\chi_A(x)\chi_B(x) = 0$ if $x \notin A \cap B$. This shows that $\chi_{A\cap B}(x) = \chi_A(x)\chi_B(x)$.

(c) $\chi_{A\cup B}(x) = 1$ if $x \in A \cup B$ and $\chi_{A\cup B}(x) = 0$ if $x \notin A \cup B$.
If $x \in A$, $\chi_A(x) + \chi_B(x) - \chi_A(x)\chi_B(x) = 1 + \chi_B(x) - \chi_B(x) = 1$.
Similarly, if $x \in B$ then $\chi_A(x) + \chi_B(x) - \chi_A(x)\chi_B(x) = 1$. So if either $x \in A$ or $x \in B$, then $\chi_A(x) + \chi_B(x) - \chi_A(x)\chi_B(x) = 1$.
This tells us that $\chi_A(x) + \chi_B(x) - \chi_A(x)\chi_B(x) = 1$ if $x \in A \cup B$.
If $x \notin A$, then $\chi_A(x) + \chi_B(x) - \chi_A(x)\chi_B(x) = \chi_B(x)$ which equals 0 if $x \notin B$.
So $\chi_A(x) + \chi_B(x) - \chi_A(x)\chi_B(x) = 0$ if x is neither in A nor B – in other words if $x \in A^c \cap B^c = (A \cup B)^c$. So $\chi_A(x) + \chi_B(x) - \chi_A(x)\chi_B(x) = 0$ if $x \notin A \cup B$.
This shows that $\chi_{A\cup B}(x) = \chi_A(x) + \chi_B(x) - \chi_A(x)\chi_B(x)$.