Problem 1.

For any propositions P and Q, write the truth tables for each of the propositions '$P \Rightarrow Q$' and '$\neg P \vee Q$'. Are these two propositions ('$P \Rightarrow Q$' and '$\neg P \vee Q$') equivalent?

Solution:

P	Q	$P \Rightarrow Q$	$\neg P$	$\neg P \vee Q$
T	T	T	F	T
F	F	T	T	T
T	F	F	F	F
F	T	T	T	T

Since '$P \Rightarrow Q$' and '$\neg P \vee Q$' have the same truth values, they must be equivalent.

Problem 2. We define a integer m to be the *smallest* number if it is smaller (lesser) than any other *integer*.

(a) Rewrite the sentence 'm is smaller than any other integer' in mathematical symbols using quantifiers if necessary.

(b) Prove that there exists no such number via a proof by contradiction.

Solution:

(a) $\forall n \in \mathbb{Z} : m < n$.

(b) If there exists such a number m, then $m < n$ for *any* $n \in \mathbb{Z}$.
So for $n = m - 1$, for instance, $m < m - 1$ – which is impossible. Contradiction.
Therefore, there exists no such number.

Problem 3. Prove by induction that $\prod_{i=2}^{n}\left(1-\frac{1}{i^2}\right)=\frac{n+1}{2n}$ for all $n \geq 2$.

Solution:

- **Base case:** For $n=2$, we have:

$$\prod_{i=2}^{\mathbf{2}}\left(1-\frac{1}{i^2}\right)=1-\frac{1}{2^2}=1-\frac{1}{4}=\frac{3}{4}=\frac{\mathbf{2}+1}{2\cdot\mathbf{2}}$$

  and so the base case holds.

- **Inductive step:** Suppose now as an inductive hypothesis that

$$\prod_{i=2}^{k}\left(1-\frac{1}{i^2}\right)=\frac{k+1}{2k}$$

  for some $k \geq 2$. Then:

$$\prod_{i=2}^{k+1}\left(1-\frac{1}{i^2}\right)=\left(\prod_{i=2}^{k}\left(1-\frac{1}{i^2}\right)\right)\cdot\left(1-\frac{1}{(k+1)^2}\right)=\frac{k+1}{2k}\cdot\frac{(k+1)^2-1}{(k+1)^2}=\frac{k^2+2k}{2k(k+1)}=\frac{k(k+2)}{2k(k+1)}$$

  and so:

$$\prod_{i=2}^{k+1}\left(1-\frac{1}{i^2}\right)=\frac{(k+1)+1}{2(k+1)}$$

  as required.

- **Conclusion:** By induction, $\prod_{i=2}^{n}\left(1-\frac{1}{i^2}\right)=\frac{n+1}{2n}$ for all $n \geq 2$.

Problem 4. Define the function $f : \mathbb{N}^2 \longrightarrow \mathbb{N}^2$ by $f(n,m) = (n+m, n-m)$.

(a) Prove that f is injective.

(b) Is f bijective? If so, prove it. If not, explain why and provide a counter-example.

Solution:

(a) Let $(n,m),(p,q) \in \mathbb{N}^2$. Then:

$$\begin{aligned} f(n,m) = f(p,q) &\Rightarrow (n+m, n-m) = (p+q, p-q) \\ &\Rightarrow n+m = p+q \text{ and } n-m = p-q \\ &\Rightarrow (n+m)+(n-m) = (p+q)+(p-q) \\ &\text{and } (n+m)-(n-m) = (p+q)-(p-q) \\ &\Rightarrow 2n = 2p \text{ and } 2m = 2q \\ &\Rightarrow n = p \text{ and } m = q \\ &\Rightarrow (n,m) = (p,q) \end{aligned}$$

Therefore, f is injective.

(b) Since f is injective, it suffices to check whether f is surjective in order to check whether it is bijective. For f to be injective, we need to check the following:

$$\forall (p,q) \in \mathbb{N}^2, \exists (n,m) \in \mathbb{N}^2 : f(n,m) = (p,q)$$

Now:

$$\begin{aligned} f(n,m) = (p,q) &\Leftrightarrow (n+m, n-m) = (p,q) \\ &\Leftrightarrow n+m = p \text{ and } n-m = q \\ &\Leftrightarrow (n+m)+(n-m) = p+q \text{ and } (n+m)-(n-m) = p-q \\ &\Leftrightarrow 2n = p+q \text{ and } 2m = p-q \end{aligned}$$

Therefore, $p-q$ and $p+q$ must both be divisible by 2.
If we choose $(p,q) = (1,0)$ then this clearly fails (since n and m wouldn't be integers from our last equation) and so f is not bijective.

Problem 5.

Let A, B and C. Prove that:

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

Solution:

$$\begin{aligned}
(x, y) \in A \times (B \cup C) &\Leftrightarrow [x \in A] \wedge [y \in (B \cup C)] \\
&\Leftrightarrow [x \in A] \wedge [[y \in B] \vee [y \in C]] \\
&\Leftrightarrow [[x \in A] \wedge [y \in B]] \vee [[x \in A] \wedge [y \in C]] \\
&\Leftrightarrow [(x, y) \in (A \times B)] \vee [(x, y) \in (A \times C)] \\
&\Leftrightarrow (x, y) \in (A \times B) \cup (A \times C)
\end{aligned}$$

Therefore, by the definition of set equality, the result follows.

Problem 6.

Define the characteristic function of a set A as being the function such that:
$\chi_A(x) = 0$ if $x \notin A$ and $\chi_A(x) = 1$ if $x \in A$.

(a) Prove that $\chi_A(x) = 1 - \chi_{A^c}(x)$ for any set A. A^c denotes the complement of the set A.

(b) Prove that $\chi_{A\cap B}(x) = \chi_A(x)\chi_B(x)$ for any sets A and B.

(c) Prove that $\chi_{A\cup B}(x) = \chi_A(x) + \chi_B(x) - \chi_A(x)\chi_B(x)$ for any sets A and B.

Solution:

(a) By definition, $\chi_{A^c}(x) = 0$ if $x \notin A^c$ and $\chi_{A^c}(x) = 1$ if $x \in A^c$.
So $\chi_{A^c}(x) = 0$ if $x \in A$ and $\chi_{A^c}(x) = 1$ if $x \notin A$.
Thus: $1 - \chi_{A^c}(x) = 1 = \chi_A(x)$ if $x \in A$ and $1 - \chi_{A^c}(x) = 0 = \chi_A(x)$ if $x \notin A$.
Conclusion: $\chi_A(x) = 1 - \chi_{A^c}(x)$.

(b) $\chi_{A\cap B}(x) = 1$ if $x \in A \cap B$ and $\chi_{A\cap B}(x) = 0$ if $x \notin A \cap B$.
Now if $x \in A$, $\chi_A(x)\chi_B(x) = \chi_B(x)$ and $\chi_B(x) = 1$ if $x \in B$. So if $x \in A$ and $x \in B$, $\chi_A(x)\chi_B(x) = 1$ and $\chi_A(x)\chi_B(x) = 0$ otherwise. In other words, if $x \in A \cap B$, $\chi_A(x)\chi_B(x) = 1$ and $\chi_A(x)\chi_B(x) = 0$ if not. Therefore, $\chi_A(x)\chi_B(x) = 1$ if $x \in A \cap B$ and $\chi_A(x)\chi_B(x) = 0$ if $x \notin A \cap B$. This shows that $\chi_{A\cap B}(x) = \chi_A(x)\chi_B(x)$.

(c) $\chi_{A\cup B}(x) = 1$ if $x \in A \cup B$ and $\chi_{A\cup B}(x) = 0$ if $x \notin A \cup B$.
If $x \in A$, $\chi_A(x) + \chi_B(x) - \chi_A(x)\chi_B(x) = 1 + \chi_B(x) - \chi_B(x) = 1$.
Similarly, if $x \in B$ then $\chi_A(x) + \chi_B(x) - \chi_A(x)\chi_B(x) = 1$. So if either $x \in A$ or $x \in B$, then $\chi_A(x) + \chi_B(x) - \chi_A(x)\chi_B(x) = 1$.
This tells us that $\chi_A(x) + \chi_B(x) - \chi_A(x)\chi_B(x) = 1$ if $x \in A \cup B$.
If $x \notin A$, then $\chi_A(x) + \chi_B(x) - \chi_A(x)\chi_B(x) = \chi_B(x)$ which equals 0 if $x \notin B$.
So $\chi_A(x) + \chi_B(x) - \chi_A(x)\chi_B(x) = 0$ if x is neither in A nor B – in other words if $x \in A^c \cap B^c = (A \cup B)^c$. So $\chi_A(x) + \chi_B(x) - \chi_A(x)\chi_B(x) = 0$ if $x \notin A \cup B$.
This shows that $\chi_{A\cup B}(x) = \chi_A(x) + \chi_B(x) - \chi_A(x)\chi_B(x)$.