Notes on Lecture 4: injections, surjections and bijections

Instructor: El Mehdi Ainasse
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This is to recapitulate what was rushed at the end of lecture.

Recall that a function $f : X \to Y$ is an **injection** if it maps **distinct inputs** to **distinct outputs**. In other words, [redacted] every $y \in Y$ is the image of **at most one** $x \in X$ – or more simply f maps at most one $x \in X$ to an element $y \in Y$. Hence why we also refer to such functions as **one-to-one** as only one $x \in X$ can be mapped to a given $y \in Y$.
This imposes a restriction on the domain: it cannot be too [redacted] If more than one input map to the same output, we'd have to throw out all but one of them to make the function injective. Therefore, restricting the domain in an appropriate way like we did $f(x) = x^2$ can turn our function into an injection.

If you think of the function $f^+ : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}, x \mapsto x^2$, then as the domain contains only non-negative numbers, each $x \geq 0$ will be the only one to map to x^2. The only other number that could square to x^2 is $-x$, but then it would be non-positive and thus outside of our domain. Example: only 2 maps to $f^+(2) = 4$. There is no other positive number that could possibly square to 4 via our function f^+.

A function is a **surjection** if every input maps to some output. In other words, if whenever $y \in Y$, there must be an $x \in X$ for which $f(x) = y$. That is to say that it is always true that, for any $y \in Y$, $f : x \mapsto y$ for some $x \in X$. If there is no $y \in Y$ to which some element of $x \in X$ maps via f, then f is not a surjection. For example, if we take $f : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$, then this cannot be a surjection because it is never true that $x^2 = -1$ for any $x \in \mathbb{R}$. (Here, our choice of y is -1.) That said, if we restricted the codomain to non-negative numbers only (so $y \geq 0$), then $x^2 = y \geq 0$ always has **at least** one solution (either $x = \sqrt{y}$ or $x = -\sqrt{y}$), making our function a surjection.
This time, this imposes a restriction on the codomain. Our codomain has to be small enough that **at least** one $x \in X$ maps to any $y \in Y$. There should be no $y \in Y$ left out from the set of possible outputs of f.

A function that is both injective and surjective is called **bijective**. If $f : X \to Y$ is injective, then (cf. above) f maps at most one $x \in X$ to any given $y \in Y$. If it is surjective, then it maps at least one $x \in X$ to any given $y \in Y$. Together, this means that a bijective function maps **exactly** one $x \in X$ to every $y \in Y$. In other words, every $x \in X$ corresponds exactly to one $y \in Y$ – which is why we also call a bijection a **one-to-one correspondence**.

This condition can be thought of as the fact that f is a bijection if and only if for all $y \in Y$, $f(x) = y$ has a unique solution. For example, the function $f : \mathbb{R} \to \mathbb{R}^{\geq 0}$ is not bijective because the equation $x^2 = y$ for $y \geq 0$ has two solutions: $x = \sqrt{y}$ and $x = -\sqrt{y}$.
Another way to see this is to notice that although f is surjective (since $f(x) = x^2 = y \geq 0$ has at least a solution, as remarked twice above), it is not injective. Indeed, every two numbers of opposite signs, for example -2 and 2 map to the same number (i.e. square to the same number); for example 4. So f is not injective. To make it injective, we can restrict the domain $\mathbb{R}$ to either $\mathbb{R}^{\geq 0}$ or $\mathbb{R}^{\leq 0}$ (which leads to the functions f^+ and f^- from the lecture). Note that it will usually be the case that we will need to restrict both the domain and the codomain to ensure that a function can be turned into a bijection – although that may not always be possible. Here is a worked out example of how to prove that a function is bijective:

Take the function $f : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (x + y, x - y)$. Let (x_1, y_1) and (x_2, y_2) be two points. Then $f(x_1, y_1) = f(x_2, y_2)$ implies that $(x_1 + y_1, x_1 - y_1) = (x_2 + y_2, x_2 - y_2)$. This tells us that: $x_1 + y_1 = x_2 + y_2$ and $x_1 - y_1 = x_2 - y_2$. Adding up these two equations term by term, we obtain that $x_1 + y_1 + x_1 - y_1 = x_2 + y_2 + x_2 - y_2$, i.e. $2x_1 = 2x_2$ and so $x_1 = x_2$. If we subtract the second euqation from the first one term by term, we obtain that $2y_1 = 2y_2$ and so $y_1 = y_2$. This tells us that $(x_1, y_1) = (x_2, y_2)$ and so f is injective.

To verify that it is surjective, let $(a, b) \in \mathbb{R}^2$ be any point and consider the equation $f(x, y) = (a, b)$. Then: $(x + y, x - y) = (a, b)$ so that $x + y = a$ and $x - y = b$ and so adding and subtracting the equations as we did above, we obtain that $x = \dfrac{a+b}{2}$ and $y = \dfrac{a-b}{2}$. This means that the equation has a solution, given by $(x, y) = \left(\dfrac{a+b}{2}, \dfrac{a-b}{2}\right)$. Indeed, we can check that $f\left(\dfrac{a+b}{2}, \dfrac{a-b}{2}\right) = (a, b)$ for any $(a, b) \in \mathbb{R}^2$ simply by plugging it in the formula for f.

We conclude that f is bijective.

Note that we could have directly used our work to prove the surjectivity of f to conclude that it is bijective since we can see that the solution is given by a unique expression. The only solution is $\left(\dfrac{a+b}{2}, \dfrac{a-b}{2}\right)$ as our work clearly showed. So we did not need to prove the injectivity of f as above. But this is not a generality. It just turns out to work nicely in this example.

Note also that the solution to that equation tells you how to map back. Indeed, if f maps (x, y) to (a, b), then we can recover (x, y) as $\left(\dfrac{a+b}{2}, \dfrac{a-b}{2}\right)$ (as seen above) – meaning that the "backwards" map to f (its inverse) is given by sending $(a, b) \in \mathbb{R}^2$ to $\left(\dfrac{a+b}{2}, \dfrac{a-b}{2}\right) \in \mathbb{R}^2$. We'll worry about this notion of an inverse later.