Problem 1.

Prove Bernoulli's inequality:

$$(1 + x)^n \geq 1 + nx$$

for all non-negative integers n and real numbers $x > -1$.

Answer:

(You may use the back of this page if necessary.)

Problem 2.

Given sets A and B, define their *symmetric difference* by:

$$A\Delta B = (A - B) \cup (B - A) = (A \cup B) - (A \cap B)$$

Prove that for any sets A, B and C, the following holds:

$$(A\Delta B)\Delta C = A\Delta(B\Delta C)$$

Answer:

(You may use the back of this page if necessary.)

Problem 3. Solve the following linear congruences:

(a) $3x \equiv 15 \mod 18$

(b) $4x \equiv 14 \mod 18$

Answer:

(You may use the back of this page if necessary.)

Problem 4.

(a) Consider the diophantine equation $98n + 35m = 13$ where n and m are integers. Does it possess any solutions? If so, prove it. If not, explain why.

(b) Solve the diophantine equation $98n + 35m = 14$ where n and m are integers.

Answer:

(You may use the back of this page if necessary.)

Problem 5.

Let n be a positive integer and $\{x_i\}$ be positive real numbers. Using the principle of induction, prove that:

$$\frac{1}{2^n}\sum_{i=1}^{2^n} x_i \geq \left(\prod_{i=1}^{2^n} x_i\right)^{1/2^n}$$

Answer:

(You may use the back of this page if necessary.)

Problem 6.

Consider the equation $x^n + y^n = z^n$.

Show that if it has a rational solution, then it has an integer solution.

(That is, show that if $x, y, z \in \mathbb{Q}$ satisfy that equation, then there are numbers $k, \ell, m \in \mathbb{Z}$ depending on x, y, z which also satisfy that equation.)

Answer:

(You may use the back of this page if necessary.)

Problem 7.

Which of the following formulae define well-defined functions $f : \mathbb{Q}^2 \to \mathbb{Q}$? If they do not, explain why.

(a) $f\left(\frac{a}{b}\right) = a + b$

(b) $f\left(\frac{a}{b}\right) = \frac{b}{a}$

(c) $f\left(\frac{a}{b}\right) = \frac{a^2}{b^2}$

Answer:

(You may use the back of this page if necessary.)