Practice Problems 2 – Solutions

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MAT 123 - Precalculus
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DUE DATE: NEVER.

Exercise 0. Review sections 3.1 – 3.5 & 4.1 – 4.4. Re-read everything thoroughly. Done? Good! You may now move to **Exercise 1**.

Exercise 1. Show that $\sqrt{2+\sqrt{3}} = \sqrt{\frac{3}{2}} + \sqrt{\frac{1}{2}}$ and that $\sqrt{9-4\sqrt{5}} = \sqrt{5} - 2$.

Solution:

$$\left(\sqrt{\frac{3}{2}} + \sqrt{\frac{1}{2}}\right)^2 = \sqrt{\frac{3}{2}}^2 + 2\cdot\sqrt{\frac{3}{2}}\cdot\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{2}}^2 = \frac{3}{2} + 2\frac{\sqrt{3}}{\sqrt{2}}\frac{\sqrt{1}}{\sqrt{2}} + \frac{1}{2} = 2 + \sqrt{3}$$

This proves the first equality.

$$(\sqrt{5}-2)^2 = \sqrt{5}^2 - 2\cdot 2\sqrt{5} + 2^2 = 5 - 4\sqrt{5} + 4 = 9 - 4\sqrt{5}$$

This proves the second equality.

Exercise 2. Solve for x:

$$\frac{\log_6(15x)}{\log_6(5x)} = 2$$

$$\ln(\log_3(e^x + 2)) = 1$$

Solution: For the first equation, we can multiply through by $\log_6(5x)$ and obtain:

$$\log_6(15x) = 2\log_6(5x)$$

$$6^{\log_6(15x)} = 6^{2\log_6(5x)} = \left(6^{\log_6(5x)}\right)^2$$

$$15x = (5x)^2 = 25x^2$$

$$15 = 25x$$

$$x = \frac{15}{25} = \frac{3}{5}$$

For the second equation:

$$\ln(\log_3(e^x + 2)) = 1$$
$$e^{\ln(\log_3(e^x+2))} = e^1 = e$$
$$\log_3(e^x + 2) = e$$
$$3^{\log_3(e^x+2)} = 3^e$$
$$e^x + 2 = 3^e$$
$$e^x = 3^e - 2$$
$$\ln(e^x) = \ln(3^e - 2)$$
$$x = \ln(3^e - 2)$$

Exercise 3. Suppose someone invested money in a bank account that compounds interest four times a year. If their current amount of money in the account is 721% times the original amount of money deposited in the account after 20 years, what was the interest rate, r, *as a percentage*?

Solution: The equation for the amount at time t is:

$$A(t) = A_0 \cdot \left(1 + \frac{r}{4}\right)^{4t}$$

Here A_0 being the initial amount.

Now we are given that at $t = 20$ years, the amount $A(t) = 721\% A_0$, and so:

$$721\% A_0 = A_0 \cdot \left(1 + \frac{r}{4}\right)^{4 \cdot 20}$$

And so:

$$7.21 A_0 = A_0 \cdot \left(1 + \frac{r}{4}\right)^{80}$$
$$7.21 = \left(1 + \frac{r}{4}\right)^{80}$$
$$\log_{80}(7.21) = \log_{80}\left(\left(1 + \frac{r}{4}\right)^{80}\right) = 1 + \frac{r}{4}$$
$$\frac{r}{4} = \log_{80}(7.21) - 1$$
$$r = 4\log_{80}(7.21) - 1 \approx 0.8032 = 80.32\%$$

(Yes, this problem is unrealistic.)

Exercise 4. Suppose a circle has a radius $r = b$ and suppose an ellipse is described by the equation $ax^2 + \frac{y^2}{b^2} = 1$ where the b is the same as the radius of the circle. What must be the value of a so that the areas of the circle and the ellipse are equal?

The equation of the ellipse needs to be put in this form:

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$$

Clearly, β is just b. So we need a to be equal to $\frac{1}{\alpha^2}$ and so $a = \frac{1}{\alpha^2}$, whence $\alpha^2 = \frac{1}{a}$, i.e.

$$\alpha = \frac{1}{\sqrt{a}}$$

Now the area of the ellipse is $\pi\alpha b = \pi \cdot \frac{1}{\sqrt{a}} \cdot b = \frac{\pi b}{\sqrt{a}}$ while the area of the circle πb^2. Therefore, for them to be equal, we need to have:

$$\pi b^2 = \frac{\pi b}{\sqrt{a}}$$

$$b = \frac{1}{\sqrt{a}}$$

We now solve for a:

$$\sqrt{a} = \frac{1}{b}$$

$$a = \frac{1}{b^2}$$

And that is the value a needs to have.

Exercise 5. Find that the number c such that the area under the graph of $1/x$ from 1 to c is equal to the *circumference* of the circle of radius $r = \sqrt{2}$.

<u>Solution:</u> The area under the graph of $1/x$ from 1 to c is equal to $\ln\left(\frac{c}{1}\right) = \ln(c)$.
The area of the circle of radius $r = \sqrt{2}$ is $\pi r^2 = \pi(\sqrt{2})^2 = 2\pi$. Therefore, we need to have $\ln(c) = 2\pi$, which means that $c = e^{2\pi}$.