Problem 1.

Simplify the following expressions:

(a) $\log_3(27\sqrt{3})$,

(b) $5^{\log_{25}(1/125)}$.

Solution:

(a)

$$\log_3(27\sqrt{3}) = \log_3(27) + \log_3(\sqrt{3}) = \log_3(3^3) + \log_3(3^{1/2}) = 3 + 1/2 = \frac{7}{2} = 3.5$$

(b)

$$5^{\log_5(1/125)} = 5^{\log_{25}\left(\frac{1}{25^2}\right)} = 5^{-\log_5(5\cdot 25)} = 5^{-\log_{25(25)} - \log_{25}(5)} = 5^{-1-\log_{25}(25^{1/2})} = 5^{-1-1/2} = 5^{-3/2}$$

Problem 2.

(a) If \$500 are invested at a 0.2% interest rate compounded continuously, what will be the amount invested after 10 years?

(b) \$1,000 are compounded continuously at a rate r. The amount after 10 years is $1000e^{0.01}$. What is r as a *percentage*?

Solution:

(a) The equation for the amount of money is:

$$a(t) = 500 \cdot (1 + 0.002)^t$$

The amount invested after 10 years is obtained by simply plugging in $t = 10$, and so the amount is $500 \cdot (1 + 0.002)^{10} \approx \$510.10$.

(b) The equation for the amount is:

$$a(t) = 1000 \cdot (1 + r)^t$$

After 10 years, the amount given is $1000e^{0.01}$ but we also know that it's the same as plugging $t = 10$ in that equation:

$$1000 \cdot (1 + r)^{10} = 1000e^{0.01}$$

$$(1 + r)^{10} = e^{0.01}$$

$$1 + r = (e^{0.01})^{1/10} = e^{0.01 \cdot (1/10)} = e^{0.001}$$

$$r = e^{0.001} - 1 \approx 0.001$$

For r as a percentage, simply multiply this by 100: $100(e^{0.001} - 1) \approx 0.1\%$.

Problem 3.

Mitosis is a process whereby a parent cell replicates its DNA to two daughter cells, therefore dividing itself into two further identical cells. (I know this looks like a biology problem, but it really isn't. Just bear with me.) Assume that, through this process, a certain colony of cells doubles its size every 4 hours. The current size of the colony at the time of measurement is 100 cells. At which point in time will it be 900? (You may leave the answer in terms of logs etc.)

Solution:

The equation for size of the colony at time t (in hours) is:

$$a(t) = 100 \cdot 2^{t/4}$$

If $a(t) = 900$ at time t, then:

$$900 = 100 \cdot 2^{t/4}$$

$$9 = 2^{t/4}$$

$$\log_2(9) = \log_2(2^{t/4}) = \frac{t}{4}$$

$$t = 4\log_2(9) \approx 12.68$$

Problem 4.

Find the area inside the ellipse $5x^2 + 2y^2 = 1$.

Solution:

The equation for an ellipse is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

In our case, we must have $\frac{1}{a^2} = 5$ and $\frac{1}{b^2} = 2$, so then: $a = \frac{1}{\sqrt{5}}$ and $b = \frac{1}{\sqrt{2}}$.

Therefore, the area is $\pi ab = \pi \cdot \frac{1}{\sqrt{5}} \cdot \frac{1}{\sqrt{2}} = \frac{\pi}{\sqrt{2}\sqrt{5}} = \frac{\pi}{\sqrt{10}}$.

Problem 5.

Solve the following equations:

(a) $e^{x^2-x} = e^{x-1}$,

(b) $e^{\sqrt{x+1}} = -1$,

(c) $\ln(x+2) = 7$.

Solution:

(a)

$$e^{x^2-x} = e^{x-1}$$
$$\ln(e^{x^2-x}) = \ln(e^{x-1})$$
$$x^2 - x = x - 1$$
$$x^2 - 2x + 1 = 0$$

Solving that whichever way you want, the solution is $x = 1$.

(b)

$$e^{\sqrt{x+1}} = -1$$

This equation has no solution because an exponential cannot be negative.

(c)

$$\ln(x+2) = 7$$
$$e^{\ln(x+2)} = e^7$$
$$x + 2 = e^7$$
$$x = e^7 - 2$$

Problem 6.

Let $f(x) = 2^{\sqrt{3x-1}} - 1$. Find the formula for the inverse function $f^{-1}(x)$.

Solution:

$$f(x) = y$$

$$2^{\sqrt{3x-1}} - 1 = y$$
$$2^{\sqrt{3x-1}} = y + 1$$
$$\log_2(2^{\sqrt{3x-1}}) = \log_2(y+1)$$
$$\sqrt{3x-1} = \log_2(y+1)$$
$$3x - 1 = (\log_2(y+1))^2$$
$$3x = (\log_2(y+1))^2 + 1$$
$$x = \frac{(\log_2(y+1))^2 + 1}{3}$$
$$\text{So } f^{-1}(x) = \frac{(\log_2(x+1))^2 + 1}{3}.$$