Problem 1.

(a) Find an equation for the line parallel to the line $4x - 2y = 5$ and passing through the point $(0, 1)$.

(b) Find an equation for the line perpendicular to the line $2y - 3x = 1$ and passing through the point $(-1, 1)$.

Solution:

(a) Let us put the given line in the form $y = mx + p$:
Since $4x - 2y = 5$, $-2y = -4x + 5$ and so $y = 2x - 5/2$. Since our line is parallel to this one, it has a slope of 2. So our line has the form $y = 2x + p$. Because it passes through the point $(0, 1)$, we have $1 = 2 \cdot 0 + p$, i.e. $p = 1$. Therefore, the equation we seek is $y = 2x + 1$.

(b) Again, same as before, we put the equation of the given line in an appropriate form: $2y - 3x = 1$ is equivalent to $2y = 3x + 1$ and so $y = (3/2)x + 1/2$. Therefore, the slope of our line is $-1/(3/2) = -(2/3)$ because it is perpendicular to this line. So our line has the form $y = -(2/3)x + p$. But since it passes through $(-1, 1)$, we have that $1 = -(2/3) \cdot (-1) + p$, so that $p = -5/3$. The equation we seek is $y = -(2/3)x - 5/3$.

Problem 2.

Consider the parabola defined by the equation $y = -x^2 - 4x - 3$.

(a) Find the coordinates of the vertex, the equation of the axis of symmetry, and the x- and y- intercepts. Make sure you simplify any expressions involving radicals.

(b) Graph the parabola indicating all the elements previously found in (a).

Solution:

(a) For graphing purposes, it is easier to complete the square.

$$y = -x^2 - 4x - 3 = -(x^2 + 4x) - 3 = -((x^2 + 2 \cdot 2x + 2^2) - 2^2) - 3 = -(x+2)^2 + 1$$

From this, we can see that the graph of this function will be similar to that of $f(x) = x^2$ except that it will be shifted to the left by 2 units, up by 1 unit and then reflected over the x-axis. So it's a parabola that is looking down.

Therefore the axis of symmetry is $x = -2$. We can also obtain this by using the formula $x = -b/2a$ for the axis symmetry and in our case $b = -4$ and $a = -1$ and so $x = -(-4)/(2 \cdot (-1)) = -2$.

The coordinates of the vertex are $x = -b/(2a)$ and $y = y(-b/(2a))$ and in our case $x = -2$ and plugging $x = -2$ in y, we get $y = -1$.

The y-intercept is the point at which $x = 0$. Plugging $x = 0$ in y, we get $y = -3$ and so the y-intercept is $(0, -3)$.

The x-intercepts are the point at which $y = 0$. We can find the x-coordinates for these points in two ways: the discriminant Δ method or by using square completion.

If we use the completion of the square, $y = 0$ is equivalent to $-(x+2)^2 + 1 = 0$, i.e. $-(x+2)^2 = -1$ and so $(x+2)^2 = 1$. Hence $x + 2 = \pm 1$. Conclusion: $x = -3$ $x = -1$.

Alternatively, if we use the discriminant method, we obtain:

$$\Delta = b^2 - 4ac = (-4)^2 - 4(-1)(-3) = 16 - 12 = 4 > 0$$

So there are two solutions:

$$x = \frac{-b + \sqrt{\Delta}}{2a} = \frac{-(-4) + 2}{2(-1)} = -3$$

Or:

$$x = \frac{-b - \sqrt{\Delta}}{2a} = \frac{-(-4) - 2}{2(-1)} = -1$$

Eitherway, the x-intercepts are $(-1, 0)$ and $(-3, 0)$.

(b) The graph will be a downward looking parabola with $(-2, -1)$ as its vertex and which crosses the x-axis at $(-1, 0)$ and $(-3, 0)$ and the y-axis at $(0, -3)$.

Problem 3.

Find the domains and ranges of each of the following functions.

You do not have to justify your answers.

(a) $f(x) = -|x-2| + 2$

(b) $f(x) = \sqrt{1-2x^2}$

(c) $f(x) = 2^{x-2}$

Solution:

(a) The domain is $(-\infty, \infty)$ because there is no division by zero happening whatsoever. Any absolute value is generally ≥ 0 and so $|x-2| \geq 0$, meaning that $-|x-2| \leq 0$. Therefore $f(x) = -|x-2| + 2 \leq 2$. The range is thus $(-\infty, 2]$.

(b) For the square root to be defined, what is inside it must be ≥ 0 and so we need for x to satisfy $1 - 2x^2 \geq 0$; i.e. $1 \geq 2x^2$ so that $x^2 \leq 1/2$ and therefore $|x| \leq 1/\sqrt{2}$. The domain is $[-1/\sqrt{2}, 1/\sqrt{2}]$. Since any square is ≥ 0 in general, the domain is just $[0, \infty)$.

(c) There is no condition on powers, so the domain is $(-\infty, \infty)$, but since 2 is a positive number, all its powers will be positive. So the range is $(-\infty, 0)$.

Problem 4.

Determine which of these functions is even and which one is odd by using the definition of an even/odd function.

(a) $f(x) = x^4\sqrt{1-4x^2}$

(b) $f(x) = x^3 - 2x$

Solution:

(a) $f(-x) = (-x)^4\sqrt{1-(-x)^2} = x^4\sqrt{1-x^2} = f(x)$ and so f is even.

(b) $f(-x) = (-x)^3 - 2(-x) = -x^3 + 2x = -(x^3 - 2x) = -f(x)$ and so f is odd.

Problem 5.

Let $f(x) = e^{2x}$ and $g(x) = \dfrac{1}{x-2}$.

Find an expression for each of the following functions: $(f \circ f)(x), (f \circ g)(x)$ and $(g \circ g)(x)$.

Solution:

$$(f \circ f)(x) = f(f(x)) = e^{2f(x)} = e^{2e^{2x}}$$

$$(f \circ g)(x) = e^{2g(x)} = e^{2\cdot(1/(x-2))} = e^{2/(x-2)}$$

$$(g \circ g)(x) = \frac{1}{g(x)-2} = \frac{1}{\frac{1}{x-2}-2} = \frac{1}{\frac{1-2(x-2)}{x-2}} = \frac{x-2}{5-2x}$$

Problem 6.

Let $f(x) = 2\sqrt{3x-1} - 1$. Find the formula for the inverse function $f^{-1}(x)$.

Solution:

Let us solve for x in terms of y in the equation $f(x) = y$:
$f(x) = y$ is equivalent to $2\sqrt{3x-1} - 1 = y$. So $2\sqrt{3x-1} = y+1$, implying that $\sqrt{3x-1} = \frac{y+1}{2}$, whence $3x - 1 = \left(\frac{y+1}{2}\right)^2 = \frac{(y+1)^2}{4}$. From there, we have that $3x = \frac{(y+1)^2}{4} + 1 = \frac{(y+1)^2+4}{4}$ and finally, we have: $x = \frac{y^2+2y+5}{12}$.

Conclusion:

$$f^{-1}(x) = \frac{x^2+2x+5}{12}$$