Nakano-Positivity of Holomorphic Hilbert Bundles and L^2 Extension Theory

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Setting

- Consider a bounded domain $\Omega \subset \mathbb{C}^n$ and a domain $U \subset \mathbb{C}^m$.
- Let φ be a weight function on $\Omega \times U$ which is smooth up to the boundary, define $\varphi_t(\cdot) := \varphi(\cdot, t)$, and let

$$L^2_{\varphi_t}(\Omega) := \left\{ f : \Omega \to \mathbb{C}; \int_\Omega |f|^2 e^{-\varphi_t} dV_\Omega < +\infty \right\} \text{ and } \mathcal{H}^2_{\varphi_t}(\Omega) := L^2_{\varphi_t}(\Omega) \cap \mathcal{O}(\Omega).$$

- Since Ω and U are bounded and φ is smooth up to the boundary on $\Omega \times U$, the Hilbert spaces $\mathcal{H}^2_{\varphi_t}(\Omega)$ are independent of t as subspaces of $\mathcal{O}(\Omega)$.
- Thus one has a vector bundle E over U whose fiber at $t \in U$ is $\mathcal{H}^2_{\varphi_t}(\Omega)$. This vector bundle is trivial as a holomorphic vector bundle, but has a non-trivial Hermitian metric given by the $L^2_{\varphi_t}$-norm on $\mathcal{H}^2_{\varphi_t}(\Omega)$ in the fiber over t.
- The goal is to find conditions under which the curvature of E is positive.

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Berndtsson's Nakano-positivity result

Theorem (Berndtsson)

If Ω is pseudoconvex and φ is (strictly) plurisubharmonic on $\Omega \times U$, then the holomorphic hermitian bundle $\left(E, \|\cdot\|_{\varphi_t}\right)$ is (strictly) Nakano-positive.

Applications of Berndtsson's theorem

Berdtsson's theorem has two important applications:

1. Convexity of the Mabuchi K-energy along geodesics in the space of Kähler metrics. (Berndtsson.)
2. Crucial tool in the proof of an optimal L^2 extension theorem for holomorphic functions. (Berndtsson-Lempert.)
   - L^2 extension is a fundamental result in complex analysis and geometry.
   - For example, useful in arguments involving induction on dimension.

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The main theorem holds (and was proved) in the setting of Stein manifolds.

- Ω a bounded pseudoconvex in a Stein manifold.
- Holomorphic functions replaced by holomorphic sections of a line bundle.
- Weights replaced by metrics for line bundle.
- The assumptions on the Hessian of φ replaced by corresponding curvature assumptions on the metrics for the line bundle.

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Sketch of proof – preliminary setup

- Let F be the bundle whose fibers are $L^2_{\varphi_t}(\Omega)$ and E be the bundle whose fibers are $\mathcal{H}^2_{\varphi_t}(\Omega)$.
- The Chern connection for F, and therefore its curvature, are easy to compute: the latter acts on an m-tuple of holomorphic sections $(u_1, \cdots, u_m)$ by

$$\sum_{1 \le j,k \le m} \left(\Theta^F_{jk} u_j, u_k\right)_{\varphi_t} = \sum_{1 \le j,k \le m} \left(\varphi_{t_j \bar{t}_k} u_j, u_k\right)_{\varphi_t}.$$

- We can view E as a subbundle of F. Therefore, the curvature of E is obtain from the curvature of F by subtracting the second fundamental form (Griffith's curvature formula):

$$(\text{Curvature of } E) = (\text{Curvature of } F) - (\text{Second Fundamental Form}).$$

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$$\sum_{1\leq j,k\leq m} \left(\Theta^F_{jk} u_j, u_k\right)_{\varphi_t} = \sum_{1\leq j,k\leq m} \left(\varphi_{t_j\bar{t}_k} u_j, u_k\right)_{\varphi_t} - \left\| \Pi_\perp \left(\sum_{1\leq j\leq m} \nabla_{t_j} u_j \right) \right\|^2_{\varphi_t}.$$

- Since $u_0 := \Pi_\perp \left(\sum_{1\leq j\leq m} \nabla_{t_j} u_j\right) = \Pi_\perp \left(\sum_{1\leq j\leq m} \varphi_{t_j} u_j\right)$ is orthogonal to the holomorphic sections, it is the minimal-norm solution of the equation

$$\bar{\partial}_z u = \bar{\partial}_z \left(\sum_{1\leq j\leq m} \varphi_{t_j} u_j \right) = \sum_{1\leq j\leq m; 1\leq \mu\leq n} \varphi_{t_j \bar{z}_\mu} u_j =: \alpha.$$

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Sketch of proof - the end

- We can now estimate the second fundamental form using Hörmander's theorem L^2-estimates for the $\bar{\partial}$-operator to obtain an upper bound for the second fundamental form, hence a lower bound for the curvature of E.
- Specifically, we have:

$$\int_\Omega |u_0|^2 e^{-\varphi_t} dV(z) \leq \int_\Omega |\alpha|^2_{\sqrt{-1}\partial\bar{\partial}\varphi} dV(z) = \int_\Omega \sum_{1\leq\mu,\nu\leq n} \varphi^{z_\mu \bar{z}_\nu} \varphi_{t_j \bar{z}_\mu} u_j \overline{\varphi_{t_k \bar{z}_\nu} u_k} e^{-\varphi_t} dV(z).$$

- All in all:

$$\sum_{1\leq j,k\leq m} \left(\Theta^F_{jk} u_j, u_k\right)_{\varphi_t} \geq \int_\Omega \left(\sum_{1\leq j,k\leq m} \varphi_{t_j \bar{t}_k} - \sum_{1\leq\mu,\nu\leq n} \varphi^{z_\mu \bar{z}_\nu} \varphi_{t_j \bar{z}_\mu} \overline{\varphi_{t_k \bar{z}_\nu}} \right) u_j \bar{u}_k e^{-\varphi_t} dV(z),$$

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L^2 extension theory

- From our knowledge of several complex variables and complex geometry, we know more or less that we can extend holomorphic objects across certain subspaces to a larger space.
- For example, we can always extend a holomorphic function or section of a line bundle from a smooth hypersurface to a complex manifold.
- The question is whether we can extend with L^2 estimates; meaning, if F extends f, do we have estimates of the sort

$$\|F\|^2 \leq C \cdot \|f\|^2 .$$

- The answer is **YES**, and such estimates can obtained with sharp constants.

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Optimal L^2 extension theorem

Theorem (Berndtsson-Lempert)

Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain and let $V \subset \Omega$ be a linear subpace of codimension m, which intersects Ω. Suppose that φ is plurisubharmonic on Ω. Then for any holomorphic function f on $V \cap \Omega$ such that

$$\int_{V\cap\Omega} |f|^2 e^{-\varphi} dA(z) < +\infty,$$

there exists a function F which is holomorphic on Ω, restrics to f on $V \cap \Omega$ and satisfies

$$\int_{\Omega} |F|^2 e^{-\varphi} dV(z) \leq \sigma_m \int_{V\cap D} |f|^2 e^{-\varphi} dV(z).$$

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Idea of proof

- Since the estimates are uniform, we can reduce to the case when Ω is a strictly pseudconvex domain with smooth boundary and then apply a limiting process.
- For $z \in \mathbb{C}^n$, write $z = (z_1, \cdots, z_m, z_{m+1}, \cdots, z_n) = (z', z'')$ and say V is defined by $z' = 0$. Let $G(z) = log\,|z'|^2$ and let $\Omega_t := \{G < t\}$ for $t \in (-\infty, 0)$. We may assume that $|z'| \leq 1$ on Ω.
- By general SCV theory, we know that f has an extension to Ω_t. Let F_t denote the extension of f of minimal L^2-norm on Ω_t.
- Proposition: The function

$$e^{-mt}\,\|F_t\|_t^2 := e^{-mt}\int_{\Omega_t} |F_t|^2\, e^{-\varphi} dV(z),$$

  is decreasing in t.
- Our goal is to obtain an estimate on $\|F_0\|_\varphi^2$.

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- The limit is in fact equal to

$$\sigma_m \int_{V\cap\Omega} |f|^2 e^{-\varphi} dA(z).$$

- We now estimate the norms $\|F_t\|_t^2$ by estimating the norms of certain measures by using Hilbert space theory:

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Demailly's approximation theorem

Theorem (Demailly)

Let φ be a plurisubharmonoc function on a bounded pseudoconvex domain $\Omega \subset \mathbb{C}^n$. For every $m > 0$, let $\mathcal{H}_{m\varphi}(\Omega)$ be the space of holomorphic functions with finite $L^2_{m\varphi}$-norm on Ω. For every m, let $\varphi_m = \frac{1}{2m} \sum_{\ell \geq 1} |\psi_\ell|^2$ where $\{\psi_\ell\}_{\ell \geq 1}$ is an orthonormal basis for $\mathcal{H}_{m\varphi}(\Omega)$. Then there are constants $C_1, C_2 > 0$ independent of m such that

$$\varphi(z) - \frac{C_1}{m} \leq \varphi_m(z) \leq \sup_{|\eta - z| < r} \varphi(\eta) + \frac{1}{m} \log \left(\frac{C_2}{r^n} \right),$$

for every $z \in \Omega$ and $r < d(z, \partial\Omega)$. In particular, φ_m converges pointwise to φ and in L^1_{loc} as m goes to $+\infty$.

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Thank you for your attention.